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# On Dimension of the Set of Solutions of A Fractional Differential Inclusion Via the Caputo-Hadamard Fractional Derivation

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**Abstract.** We investigate the existence of solution for a fractional integro-differential inclusion via the Caputo-Hadamard fractional derivation. We prove that dimension of the set of solutions for the inclusion problem is infinite dimensional under some conditions.

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**Keywords and Phrases:** Fractional integro-differential inclusion, dimension of the set of solutions, the Caputo-Hadamard fractional derivation

## 1. Introduction

One can find basic notions of fixed point theory and fractional differential theory in some related books (see for examples, [3], [27], [31], [32], [35]). There are a lot of published papers on fractional differential equations (see for example, [1], [2], [4], [7], [10], [12]-[18], [24], [25], [29], [33], [34], [36]-[40]) and inclusions ([6], [8], [9], [11], [20], [21]). Let $(Y, \rho)$

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be a metric space. Denote the class of all nonempty, closed, compact, convex and compact subsets of $Y$ by $2^Y$, $P_{cl}(Y)$, $P_{cp}(Y)$ and $P_{cp,cv}(Y)$ respectively. We say that a map $T : Y \to 2^Y$ has a fixed point if there is $y \in Y$ such that $y \in Ty$. We say that $T : Y \to P_{cl}(Y)$ is lower semi-continuous whenever $T^{-1}(A) := \{y \in Y : Ty \cap A \neq \emptyset\}$ is open for each open set $A$ of $Y$. Also, $T$ is called upper semi-continuous whenever $\{y \in Y : Ty \subset B\}$ is open for every open set $B$ of $Y$. A multifunction $T : Y \to P_{cp}(Y)$ is compact whenever $\overline{T(M)}$ is compact for all bounded subset $M$ of $Y$. Also, $T : I \to P_{cl}(\mathbb{R})$ is called measurable if $t \mapsto dis(y, T(t)) = \inf\{|y - z| : z \in T(t)\}$ is a measurable function for all $y \in \mathbb{R}$, where $I = [1, e]$. The Pompeiu-Hausdorff metric $H : 2^Y \times 2^Y \to [0, \infty)$ is defined by $H(D, G) = \max\{\sup_{d \in D} \rho(d, G), \sup_{g \in G} \rho(D, g)\}$, where $\rho(D, g) = \inf_{d \in D} \rho(d, g)$ ([19]). Then, $(P_{bd,cl}(Y), H)$ is a metric space while $(P_{cl}(Y), H)$ is a generalized metric space ([19]). We say that $T : Y \to 2^Y$ is a contraction if there is $\gamma \in (0, 1)$ such that $H(T(y), T(y')) \leqslant \gamma \rho(y, y')$ for all $y, y' \in Y$. Nadler and Covitz showed that every closed valued contractive multi-valued map has a fixed point on a complete metric space ([22]). A multi-valued map $T : I \times \mathbb{R}^2 \to 2^{\mathbb{R}}$ is said to be Caratheodory if $t \mapsto T(t, x_1, x_2)$ is measurable for all $x_1, x_2 \in \mathbb{R}$ and $(x_1, x_2) \mapsto T(t, x_1, x_2)$ is upper semi-continuous for almost all $t \in I$ ([23] and [28]). A Caratheodory multi-valued map $T : I \times \mathbb{R}^2 \to 2^{\mathbb{R}}$ is said to be $L^1$-Caratheodory if for every $\delta > 0$ there is $\phi_\delta \in L^1(I, \mathbb{R}^+)$ so that $\| T(t, x_1, x_2) \| = \sup\{|w| : w \in T(t, x_1, x_2)\} \leqslant \phi_\delta(t)$ for all $|x_1|, |x_2| \leqslant \delta$ and for almost all $t \in I$ ([23] and [28]). As you know, the Hadamard fractional integral of order $\beta > 0$ for a map $g$ is defined by $I_b^\beta g(t') = \frac{1}{\Gamma(\beta)} \int_b^{t'} (\ln \frac{t'}{s})^{\beta - 1} \frac{g(s)}{s} ds$, where $b > 0$ and $t' > b$ ([26]). In particular, we have $I_1^\beta g(t) := I^\beta g(t)$. Let $n \geqslant 1$, $0 < a < b < \infty$, $n - 1 < \beta < n$ and $g \in AC_\delta^n[a, b]$, where $AC_\delta^n[a, b] = \{g : [a, b] \to \mathbb{R} : \delta^{n-1} g(t) \in AC[a, b], \delta = t\frac{d}{dt}\}$. The Caputo-Hadamard fractional derivative is defined by ${}_H^C D_a^\beta g(t) = \frac{1}{\Gamma(n - \beta)} \int_a^t (\ln \frac{t}{s})^{n - \beta - 1} \delta^n \frac{g(s)}{s} ds := I_a^{n-\beta} \delta^n g(t)$ ([26]). Also, the Caputo-Hadamard fractional derivative of order $n$ is defined by ${}_H^C D_a^n g(t) = \delta^n g(t)$ ([26]). In particular, ${}_H^C D_1^0 g(t) = g(t)$ and ${}_H^C D_1^\alpha g(t) := {}_H^C D^\alpha g(t)$ for all $t$ ([26]). Let $\beta > 0$, $n = [\beta] + 1$ and $\alpha > 0$. Then, we have ${}_H^C D_a^\beta (\ln \frac{t}{a})^k = 0$ for $k = 0, 1, ..., n - 1$ and ${}_H^C D_a^\beta (\ln \frac{t}{a})^{\alpha - 1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta)} (\ln \frac{t}{a})^{\alpha - \beta - 1}$ for $\alpha > n$. Also, ${}_H^C D_a^\beta c = 0$ for all

$c \in \mathbb{R}$ ([26]). Let $n \geqslant 1$, $n-1 < \beta < n$ and $g \in AC^n_\delta[a,b]$. Then, $I^\beta_a({}^C_H D^\beta_a)g(t) = g(t) + \sum_{i=0}^{n-1} k_i(\ln \frac{t}{a})^i$ for some $k_0, k_1, ..., k_{n-1} \in \mathbb{R}$. Also, ${}^C_H D^\beta_a(I^\beta_a)g(t) = g(t)$ ([26]).

In 2013, Baleanu, Mohammadi and Rezapour studied the nonlinear fractional differential equation $D^\alpha u(t) = f(t, u(t))$ ($t \in I = [0,T]$, $0 < \alpha < 1$) via the periodic boundary condition $u(0) = 0$, where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous increasing function and ${}^cD^\alpha$ denotes the Caputo fractional derivative of order $\alpha$ ([15]). In 2015, Agarwal, Baleanu, Hedayati and Rezapour reviewed the existence of solution for the Caputo fractional differential inclusion ${}^cD^q x(t) \in F(t, x(t), {}^c D^\beta x(t))$ via the boundary value conditions $x(1) + x'(1) = \int_0^\eta x(s)ds$ and $x(0) = 0$, where $0 < \eta < 1$, $1 < q \leqslant 2$, $0 < \beta < 1$ and $q - \beta > 1$ ([7]). The aim of this work is to study the existence of solution for the fractional integro-differential inclusion

$${}^C_H D^\alpha x(t) \in F(t, x(t), I^\beta x(t)), \tag{1}$$

with boundary values $x(1) = g(e, x(e))$, where $0 < \alpha < 1$, $\beta > 0$, $F : I \times \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is multifunction under some conditions and $g : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a map. Also, we show that $\mathcal{S}$ is infinite dimensional under some conditions, where $\mathcal{S}$ is the set of solutions of the problem. We need next results.

**Lemma 1.1.** *([30]) Let $Q$ be a Banach space, $T : I \times Q \to P_{cp,cv}(Q)$ an $L^1$-Caratheodory multi-valued function and $A : L^1(I,Q) \to C(I,Q)$ a linear continuous map. Then, the map $AoS_T : C(I,Q) \to P_{cp,cv}(C(I),Q)$ defined by $(AoS_T)(x) = A(S_{T,x})$ is closed graph.*

**Lemma 1.2.** *[5] Let $T : [1,e] \to P_{cp,cv}(\mathbb{R}^n)$ be measurable so that $\mu(\{t : \dim T(t) < 1\}) = 0$, where $\mu$ is the Lebesgue measure. Then there exist linearly independent measurable selections $s_1(.), s_2(.), ..., s_m(.)$ of $T$ for all $m \geqslant 1$.*

**Lemma 1.3.** *[5] Let $D$ be convex and closed subset of a Banach space $Q$ and $T : D \to P_{cp,cv}(D)$ a $\delta$-contraction. If $\dim T(t) \geqslant n$ for all $t \in D$, then $\dim Fix(T) \geqslant n$.*

## 2. Main Results

Let $w \in C(I, \mathbb{R}^n)$, $\beta \in (0,1)$ and $\alpha > 0$. Consider the fractional problem ${}^{C}_{H}D^{\alpha}x(t) = w(t)$ with the boundary conditions $x(1) = g(e, x(e))$. Then, the unique solution of the problem is given by

$$x(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w(s)}{s} ds + g(e, x(e)),$$

(see [26]). We say that $x \in C(I, \mathbb{R}^n)$ is a solution for the problem (1) if it satisfies the boundary condition and there is $w \in L^1(I, \mathbb{R}^n)$ such that $w(t) \in F(t, x(t), I^{\beta}x(t))$ for almost all $t \in I$ and $x(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w(s)}{s} ds + g(e, x(e))$. The Banach space $Y = C([1, e], \mathbb{R}^n)$ is endowed with the norm $\|h\| = \sup_{s\in I} |h(s)|$. The set of selections of $F$ at $x$ is denoted by

$$S_{F,x} := \{w \in L^1(I, \mathbb{R}^n) : \ w(t) \in F(t, x(t), I^{\beta}x(t)) \text{ for almost all } t \in I\}$$

for all $x \in X$.

**Theorem 2.1.** *Let $m, p \in C(I, \mathbb{R}^+)$ be such that $l = \frac{\|m\|}{\Gamma(\beta+1)}(1 + \frac{1}{\Gamma(\alpha+1)}) + \|p\| < 1$. Assume that $F : I \times \mathbb{R}^n \times \mathbb{R}^n \to P_{cv,cp}(\mathbb{R}^n)$ is a multi-valued function such that the map $t \vdash F(t, x_1, x_2)$ is measurable and $H(F(t, x_1, x_2), F(t, y_1, y_2)) \leqslant m(t) \sum_{i=1}^{2}(|x_i - y_i|)$ and $g : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a map such that $|g(t, x) - g(t, y)| \leqslant p(t)|x - y|$ for almost all $t \in I$ and $\in x_1, x_2, y_1, y_2, x, y \in \mathbb{R}^n$. Then the inclusion problem (1) has a solution.*

**Proof.** Since $t \vdash F(t, x(t), I^{\alpha}x(t))$ is closed valued and measurable for all $x \in Y$, $S_{F,x}$ is nonempty. Define $M : X \to 2^X$ by

$$M(x) = \Big\{h \in Y : \exists \ w \in S_{F,x} \text{ s. t. } h(t) = w(t) \text{ for all } t \in I\Big\},$$

where $w(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{v(s)}{s} ds + g(e, x(e))$ for all $t \in I$. We show that $M(x)$ is closed for all $x \in Y$. Let $x \in Y$ and $\{u_n\}_{n\geqslant1}$ be a sequence in $Y(x)$ with $u_n \to u$. For every $n \geqslant 1$, choose $w_n \in S_{F,x}$ such that $u_n(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w_n(s)}{s} ds + g(e, x(e))$ for almost all $t \in I$. Since $F$ has compact values, $\{w_n\}_{n\geqslant1}$ has a subsequence which converges to some

$w \in L^1(I, \mathbb{R})$. Denote the subsequence again by $\{w_n\}_{n \geqslant 1}$. One can check that $w \in S_{F,x}$ and $u_n(t) \to u(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w(s)}{s} ds + g(e, x(e))$ for all $t \in I$. Hence, $w \in M(x)$. Thus, $M$ has closed values. Now, we show that $M$ is contractive with constant $l = \frac{\|m\|}{\Gamma(\beta+1)}(1 + \frac{1}{\Gamma(\alpha+1)}) + \|p\| < 1$. Let $x, y \in Y$ and $h_1 \in M(y)$. Choose $w_1 \in S_{F,y}$ such that $h_1(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w_1(s)}{s} ds + g(e, y(e))$ for almost all $t \in I$. Since

$$H\Big(F(t, x(t), I^\alpha x(t)), F(t, y(t), I^\alpha y(t))\Big)$$

$$\leqslant m(t)\Big(|x(t) - y(t)| + |I^\alpha x(t) - I^\alpha y(t)|\Big),$$

for almost all $t \in I$, there exists $w_1' \in (F(t, x(t), I^\alpha x(t))$ such that

$$|w_1(t) - w_1'| \leqslant m(t)\Big(|x(t) - y(t)| + |I^\alpha x(t) - I^\alpha y(t)|\Big),$$

for almost all $t \in I$. Define the multifunction $U_1 : I \to 2^{\mathbb{R}^n}$ by

$$U_1(t) = \Big\{w' \in \mathbb{R}^n : |v_1(t) - w'| \leqslant m(t)\Big(|x(t) - y(t)| + |I^\alpha x(t) - I^\alpha y(t)|\Big) \text{ for almost all } t \in I\Big\}.$$

One can check that $U_1(.) \bigcap (F(., x(.), I^\alpha x(t), I^\alpha y(.))$ is measurable.
Choose $w_2 \in S_{F,x}$ such that

$$|w_1(t) - w_2(t)| \leqslant m(t)\Big(|x(t) - y(t)| + |I^\alpha x(t) - I^\alpha y(t)|\Big),$$

for almost all $t \in I$. Now, consider $h_2 \in M(x)$ which is defined by

$$h_2(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w_2(s)}{s} ds + g(e, x(e)).$$

Hence, we get

$$|h_1(t) - h_2(t)| \leqslant \frac{1}{\Gamma(\beta)} \int_1^t (t-s)^{\beta-1} |w_1(s) - w_2(s)| ds + |g(e, x(e)) - g(e, y(e))|$$

$$\leqslant \left( \frac{\|m\|}{\Gamma(\beta+1)} (1 + \frac{1}{\Gamma(\alpha+1)}) + \|p\| \right) \|x - y\|,$$

and so $\|h_1 - h_2\| \leqslant \left( \frac{\|m\|}{\Gamma(\beta+1)} (1 + \frac{1}{\Gamma(\alpha+1)}) + \|p\| \right) \|x-y\| = l\|x-y\|$. Thus, $M$ is a contraction with closed values and so has a fixed point $x_0$. It is easy to check that $x_0$ is a solution for the inclusion problem (1). □

**Lemma 2.2.** *Assume that $z \in C(I, \mathbb{R}^+)$ and $T : I \times \mathbb{R}^n \times \mathbb{R}^n \to P_{cv,cp}(\mathbb{R}^n)$ is a multi-valued function such that $t \vdash T(t, x_1, x_2)$ is measurable and*

$$\|T(t, x_1, x_2)\| = \sup\{|w| : w \in T(t, x_1, x_2)\} \leqslant z(t)$$

*for almost all $t \in I$ and $\in x_1, x_2 \in \mathbb{R}^n$. Define $G_1 : X \to P(X)$ by*

$$G_1(x) = \left\{ k \in Y : \exists\ w \in S_{T,x} \text{ such that } k(t) = s(t) \text{ for all } t \in I \right\},$$

*where $s(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w(s)}{s} ds + g(e, x(e))$. Then $G_1(x) \in P_{cp.cv}(X)$ for all $x \in X$.*

**Proof.** Note that, $G_1 = \theta \circ S_T$, where $\theta : L^1(I, \mathbb{R}^n) \to Y$ is the continuous map defined by $\theta v(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{v_2(s)}{s} ds + g(e, x(e))$ (see Lemma 1.1). Let $x \in Y$ and $\{w_n\}$ a sequence in $S_{T,x}$. Then, $w_n(t) \in T(t, x(t), I^\alpha x(t))$ for almost $t \in I$. Since $T(t, x(t), I^\alpha x(t))$ is compact for all $t \in I$, we can choose a convergent subsequence of $\{w_n(t)\}$ (denote it again by $\{w_n(t)\}$) which converges in measure to some $w \in S_{T,x}$. Since $\theta$ is continuous, $\theta w_n(t) \to \theta w(t)$ pointwise on $I$. For showing uniform convergence, we show that $\{\theta w_n\}$ is equi-continuous. For $\tau < t \in I$, we have

$$|\theta w_n(t) - \theta w_n(\tau)| =$$

$$|\frac{1}{\Gamma(\beta)} \int_1^t (\ln(\frac{t}{s}))^{\beta-1} \frac{w_n(s)}{s} ds - \frac{1}{\Gamma(\beta)} \int_1^\tau (\ln(\frac{t}{s}))^{\beta-1} \frac{w_n(s)}{s} ds| \leqslant$$

$$|\frac{1}{\Gamma(\beta)} \int_1^\tau (\ln(\frac{t}{s}))^{\beta-1} - (\ln(\frac{\tau}{s}))^{\beta-1}) \frac{w_n(s)}{s} ds| + |\frac{1}{\Gamma(\beta)} \int_\tau^t \ln(\frac{t}{s}))^{\beta-1} \frac{w_n(s)}{s} ds|.$$

This shows that $\{\theta w_n\}$ is equi-continuous and by using the Arzela-Ascoli theorem, there is a uniformly convergent subsequence (we show it again by $\{w_n\}$) such that $\theta w_n \to \theta w$. Note that, $\theta w \in \theta(S_{T,x})$. Hence, $G_1 x = \theta(S_{T,x})$ is compact for all $x \in Y$. Now, we prove that $G_1 x$ is convex for all $x \in Y$. For $h, h' \in G_1 x$, there are $w, w' \in S_{T,x}$ such that $h(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w(s)}{s} ds + g(e, x(e))$ and $h'(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{w'(s)}{s} ds + g(e, x(e))$ for almost all $t \in I$. Let $0 \leqslant \lambda \leqslant 1$. Then, $\lambda h(t) + (1-\lambda) h'(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{(\lambda w(s) + (1-\lambda) w'(s))}{s} ds$ for almost all $t$. Since $S_{T,x}$ is convex (because $T$ has convex values), $\lambda h + (1-\lambda) h' \in G_1 x$. $\square$

Now, we provide application of our last results. In fact, it is about $dim\mathcal{S}$. It is well-known that $Fix G_1 = \mathcal{S}$.

**Theorem 2.3.** *Assume that $m, p \in C(I, \mathbb{R}^+)$ and $T : I \times \mathbb{R}^2 \times \mathbb{R}^2 \to P_{cv,cp}(\mathbb{R}^2)$ is a multifunction and so that $H(T(t, x_1, x_2), T(t, y_1, y_2)) \leqslant m(t) \sum_{i=1}^{2} |x_i - y_i|$, the map $t \vdash T(t, x_1, x_2)$ is measurable, $\|T(t, x_1, x_2)\| = \sup\{|v| : v \in T(t, x_1, x_2)\} \leqslant m(t)$ and $g : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a map such that $|g(t, x) - g(t, y)| \leqslant p(t)|x - y|$ for almost all $t \in I$ and $x_1, x_2, y_1, y_2, x, y \in \mathbb{R}^n$. If $\mu(\{t : \dim T(t, x_1, x_2) < 1$ for some $x_1, x_2 \in \mathbb{R}^n\}) = 0$ and $l := \frac{\|m\|}{\Gamma(\beta+1)}(1 + \frac{1}{\Gamma(\alpha+1)}) + \|p\| < 1$, then $dim\mathcal{S} = \infty$.*

**Proof.** Again consider the operator $G_1$ in last result. By using Lemma 2.2, $G_1 x \in P_{cp,cv}(Y)$ for all $x \in Y$. Similar to proof of Theorem 2.1, we can show that $G_1$ is contraction. Let $x \in Y$, $m \geqslant 1$ and $G'(t) = T(t, x(t), I^\alpha x(t))$ for all $t$. By using Lemma 1.2, there exist linearly independent measurable selections $v_1(.), v_2(.), ..., v_m(.)$ of $G'$. Put $h_i(t) = \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} \frac{v_i(s)}{s} ds + g(e, x(e))$ for all $1 \leqslant i \leqslant m$. If $\sum_{i=1}^{m} a_i h_i(t) = 0$ for almost $t \in I$, by using the Caputo-Hadamard derivative we get $\sum_{i=1}^{m} a_i v_i(t) = 0$ for almost $t \in I$. Hence $a_i = 0$ for all $1 \leqslant i \leqslant m$. This implies that $h_i$ are linearly independent and so $\dim G_1 x \geqslant m$. By using Lemma 1.3, we conclude that $dim\mathcal{S} = \infty$. $\square$

## Competing interests

The authors declare that they have no competing interests.


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## contributions

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