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# Another Form of Supra Ordered Separation Axioms

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**Abstract.** In this work, we present and study new ordered separation axioms, namely supra $T_{c_i}$-ordered spaces (briefly, $ST_{c_i}$-ordered spaces), where $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$. With the help of examples, we illustrate the relationships among these ordered spaces and point out under what conditions they are hereditary properties. Also, we derive some results which associate some of $ST_{c_i}$-ordered spaces with some topological notions such as supra limit points and supra disconnected spaces, and with some algebraic notions such as largest and smallest elements. Furthermore, we investigate the image of theses ordered spaces under $S^\star$-homeomorphism maps. Finally, we verify that the finite ordered product of $ST_{c_i}$-ordered spaces is $ST_{c_i}$-ordered for $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}$.

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## 1. Introduction

In 1965, Nachbin [22] originated the notion of topological ordered spaces by defining a partial order relation $\preceq$ and a topology on a non empty set $X$. He studied and discussed many properties of monotone sets and topological ordered spaces. McCartan [20], in 1968, utilized monotone neighbourhoods and monotone open sets to present $T_i$-ordered spaces and strong $T_i$-ordered spaces $(i = 0, 1, 2, 3, 4)$, respectively. He investigated the equivalent conditions for each one of these spaces and gave several examples to show that these spaces are strictly stronger than $T_i$-spaces in general topology. In 1971, McCartan [21] defined the concepts of continuous, anti-continuous and bicontinuous for topological ordered spaces and investigated their characteristics.

Arya and Gupta [10] defined and studied semi $T_0$-ordered and semi $T_1$-ordered spaces in 1991, and Leela and Balasubramanian [18] introduced and discussed $\beta T_0$-ordered and $\beta T_1$-ordered spaces in 2002. Das [12], in 2004, established a concept of supra topological ordered spaces and formulated some ordered separation axioms. Kumar [17] introduced the concepts of continuous and homeomorphism maps between topological ordered spaces.

In 1983, Mashhour et al. [19] introduced supra topological spaces by dropping only the intersection condition. They introduced the concepts of $ST_i$-spaces $(i = 0, 1, 2, 2\frac{1}{2})$ and derived their fundamental features. El-shafei et al. [13] utilized monotone supra open sets to initiate new ordered separation axioms, namely $SST_i$-ordered spaces $(i = 0, 1, 2, 3, 4)$. They also verified that $SST_i$-ordered spaces and $ST_i$-ordered spaces are equivalent in the cases of $i = 0, 1$. Abo-elhamayel and Al-shami [1] established and investigated the concepts of x-supra continuous, x-supra open, x-supra closed and x-supra homeomorphism maps between supra topological ordered spaces. Recently, some ordered maps are defined and discussed based on some celebrated generalized supra open sets (see, for example,[4, 5, 9, 11, 14, 15]). Recently, [7] explored separation axioms on supra soft topological spaces and [6] defined the concept of supra soft topological spaces.

Ordered separation axioms are well known notions and powerful tools on ordered topology and supra ordered topology. So this study focus on defining new ordered separation axioms, namely $ST_{c_i}$-ordered spaces ($i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$). We illustrate the relationships among these ordered spaces and show that $ST_{c_i}$-ordered spaces are stronger than $SST_i$-ordered spaces in case of $i = 1, 2$, and are stronger than $ST_i$-spaces in case of $i = 0, 1, 2$. Also, we derive some interesting results which connect some of the initiated ordered spaces with some topological notions such as supra limit points, supra disconnected spaces and $S^{\star}$-homeomorphism maps, and with some algebra notions such as largest and smallest elements. Moreover, we point out under what conditions the $ST_{c_i}$-ordered spaces are hereditary properties and provide several examples to elucidate the main findings obtained. In the end, we present some results of monotone sets and verify that the finite ordered product of $ST_{c_i}$-ordered spaces is $ST_{c_i}$-ordered, for $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}$.

## 2. Preliminaries

We recall in this section, some definitions and results which we need in the sequel.

**Definition 2.1.** [16] *A binary relation $\preceq$ is called a partial order relation if it is reflexive, anti-symmetric and transitive.*

**Definition 2.2.** [16] *The usual partial order relation on the set of real numbers $\mathbb{R}$ is defined as follows $\preceq = \{(a, b) : a \leqslant b, a, b \in \mathbb{R}\}$.*

**Definition 2.3.** [16] *Let $(X, \preceq)$ be a partially ordered set. An element $a \in X$ is called:*

**(i)** *A smallest element of $X$ provided that $a \preceq x$ for all $x \in X$.*

**(ii)** *A largest element of $X$ provided that $x \preceq a$ for all $x \in X$.*

**(iii)** *A minimal element of $X$ provided that $x \preceq a$ implies that $x = a$.*

**(iv)** *A maximal element of $X$ provided that $a \preceq x$ implies that $x = a$.*

**Definition 2.4.** [16] *A map* $f : (X, \preceq_1) \to (Y, \preceq_2)$ *is called order embedding if* $a \preceq_1 b$ *if and only if* $f(a) \preceq_2 f(b)$*, for each* $a, b \in X$*.*

**Definition 2.5.** [19] *A family* $\mu$ *of subsets of a non empty set* $X$ *is called a supra topology provided that the following two conditions hold.*

**(i)** $X$ *and* $\emptyset \in \mu$*.*

**(ii)** $\mu$ *is closed under arbitrary union.*

Then the pair $(X, \mu)$ is called a supra topological space. Every element of $\mu$ is called a supra open set and its complement is called a supra closed set.

**Definition 2.6.** [19] *A map* $f : X \to Y$ *is called* $S^{\star}$*-continuous if the inverse image of each supra open subset of* $Y$ *is a supra open subset of* $X$*.*

**Definition 2.7.** [19] *Any property which when satisfied by a supra topological space is also satisfied by every subspace of this supra topology is called a hereditary property.*

**Definition 2.8.** [22] *Let* $E$ *be a subset of a partially ordered set* $(X, \preceq)$ *and* $x \in X$*. Then:*

**(i)** $i(x) = \{a \in X : x \preceq a\}$ *and* $d(x) = \{a \in X : a \preceq x\}$*.*

**(ii)** $i(E) = \bigcup\{i(e) : e \in E\}$ *and* $d(E) = \bigcup\{d(e) : e \in E\}$*.*

**(iii)** *A set* $E$ *is called increasing (resp. decreasing), If* $E = i(E)$*(resp.* $E = d(E)$*).*

**Definition 2.9.** [12] *A triple* $(X, \tau, \preceq)$ *is said to be a supra topological ordered space, where* $(X, \mu)$ *is a supra topological space and* $(X, \preceq)$ *is a partially ordered set.*

**Definition 2.10.** [22] *Let* $A$ *be a subset of a topological ordered space* $(X, \tau, \preceq)$*. We define a topological ordered subspace* $(A, \tau_A, \preceq_A)$ *of* $(X, \tau, \preceq)$ *as follows* $\tau_A$ *is a relative supra topology of* $\tau$ *and* $\preceq_A = \preceq \bigcap A \times A$*.*

**Proposition 2.11.** [22] *If $U$ is an increasing (resp. a decreasing) subset of a partially ordered set $(X, \preceq)$, then $U \bigcap A$ is an increasing (resp. a decreasing) subset of a partially ordered set $(A, \preceq_A)$.*

**Definition 2.12.** [13] *A supra topological ordered space $(X, \mu, \preceq)$ is called:*

**(i)** *Lower $SST_1$-ordered space if for each $a \not\preceq b$ in $X$, there exists an increasing supra open set $G$ containing $a$ such that $b$ belongs to $G^c$.*

**(ii)** *Upper $SST_1$-ordered space if for each $a \not\preceq b$ in $X$, there exists a decreasing supra open set $G$ containing $b$ such that $a$ belongs to $G^c$.*

**(iii)** *$SST_0$-ordered if it is lower $SST_1$-ordered or upper $SST_1$-ordered.*

**(iv)** *$SST_1$-ordered if it is both lower $SST_1$-ordered and upper $SST_1$-ordered.*

**(v)** *$SST_2$-ordered if for every $a, b \in X$ such that $a \not\preceq b$, there exist disjoint supra open sets $G$ and $H$ containing $a$ and $b$, respectively, such that $G$ is increasing and $H$ is decreasing.*

**Definition 2.13.** [23] *A subset $E$ of supra topological ordered space $(X, \mu)$ is called an m-set if $E \bigcap G$ is a supra open set, for each $G \in \mu$.*

**Definition 2.14.** [2] *Let $(X, \mu)$ be a supra topological space and $x \in X$. A point $x$ is said to be supra limit point of $A \subseteq X$ provided that every supra neighborhood of $x$ contains at least one point of $A$ different than $x$. We denote all supra limit points of $A$ by $A^{sl}$.*

**Proposition 2.15.** [2] *Let $A$ and $B$ be subsets of $(X, \mu)$. Then*

**(i)** *If $A \subseteq B$, then $A^{sl} \subseteq B^{sl}$.*

**(ii)** $A^{sl} \bigcup B^{sl} \subseteq (A \bigcup B)^{sl}$.

**(iii)** *$A$ is supra closed if and only if $A^{sl} \subseteq A$.*

From now on, the set of real numbers and the diagonal relation on any non-empty set shall be briefly denoted by $\mathbb{R}$ and $\triangle$, respectively.

## 3. New Separation Axioms in Supra Topological Ordered Spaces

This section is devoted to defining and studying $ST_{c_i}$-ordered spaces ($i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$). We provide several examples to show the relationships among them. In general, we investigate some results which associate them with some topological concepts such as supra limit points, supra disconnected spaces and $S^{\star}$-homeomorphism maps. Finally, we prove that $ST_{c_i}$-ordered spaces are preserved under finite product space in the cases of $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}$.

**Definition 3.1.** *A supra topological ordered space $(X, \mu, \preceq)$ is called $ST_{c_0}$-ordered if for every $a \npreceq b$ in $X$, there exist an increasing supra open set $G$ containing $a$ such that $b \in G^c$ or a decreasing supra open set $H$ containing $b$ such that $a \in H^c$.*

**Example 3.2.** Let $\mu = \{\emptyset, G \subseteq \mathbb{R}$ such that $1 \in G$ or $2 \in G\}$ be a supra topology on $\mathbb{R}$ and $\preceq = \triangle \bigcup \{(1, 2)\}$ be a partial order relation on $\mathbb{R}$. Then $(X, \mu, \preceq)$ is an $ST_{c_0}$-ordered space.

**Example 3.3.** Let $\mu = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$ be a supra topology and $\preceq = \triangle \bigcup \{(b,\ a), (a, c)(b, c)\}$ be a partial order relation on $X = \{a, b, c\}$. Since $a \npreceq b$, then $i(a) = \{a, c\} \subseteq X$ and $d(b) = \{b\} \subseteq d(\{b, c\}) = X$. Hence $(X, \mu, \preceq)$ is not an $ST_{c_0}$-ordered space.

**Proposition 3.4.** *Every $SST_0$ -ordered space $(X, \mu, \preceq)$ is $ST_{c_0}$-ordered.*

**Proof.** It is clear. $\square$

The converse of the above proposition need not be true in general as shown in the following example.

**Example 3.5.** Assume that $(X, \mu, \preceq)$ is the same as in Example 3.2. We can observe the following two cases:

**(i)** $1 \npreceq 3$ and there is no a decreasing supra open set containing 3 which does not contain 1. Hence $(X, \mu, \preceq)$ is not an upper $SST_1$-ordered space.

**(ii)** $3 \npreceq 2$ and there is no an increasing supra open set containing 3 which does not contain 2. Hence $(X, \mu, \preceq)$ is not a lower $SST_1$-ordered space.

So from the above two cases, we find that $(X, \mu, \preceq)$ is not an $SST_1$-ordered space.

**Definition 3.6.** *A supra topological ordered space $(X, \mu, \preceq)$ is called supra continuous if for any supra open subset $G$ of $X$, $i(G)$ and $d(G)$ are supra open sets.*

**Theorem 3.7.** *Let a supra topological ordered space $(X, \mu, \preceq)$ be supra continuous. Then $(X, \mu, \preceq)$ is $ST_{c0}$-ordered if and only if $scl(d(a)) \neq scl(d(b))$ or $scl(i(a)) \neq scl(i(b))$, for each $a, b \in X$.*

**Proof.** *Necessity*: Suppose $(X, \mu, \preceq)$ is an $ST_{c0}$-ordered space and let $a \npreceq b$. Then we have two cases:

**(i)** there exists an increasing supra open set $G$ containing $a$ such that $b \in G^c$. Then $scl(d(b)) \subseteq G^c$. Therefore $a \notin scl(d(b))$. Obviously, $a \in scl(d(a))$. Thus $scl(d(a)) \neq scl(d(b))$.

**(ii)** or a decreasing supra open set $H$ containing $b$ such that $a \in H^c$. Similarity, we find that $scl(i(a)) \neq scl(i(b))$.

*Sufficiency*: Suppose that $scl(d(a)) \neq scl(d(b))$. It is well known that $a \npreceq b$ or $b \npreceq a$. Say, $a \npreceq b$. Then there exists $x \in scl(d(a))$ and $x \notin scl(d(b))$ or $x \in scl(d(b))$ and $x \notin scl(d(a))$. Say, $x \in scl(d(a))$ and $x \notin scl(d(b))$. Therefore there exists a supra open set $G$ containing $x$ such that $G \bigcap d(b) = \emptyset$. Now, $i(G) \bigcap \{b\} = \emptyset$. Since $(X, \tau, \preceq)$ is supra continuous, then $i(G)$ is a supra open set. Since $i(G) \bigcap d(a) \neq \emptyset$, then $a \in i(G)$. Thus $(X, \mu, \preceq)$ is an $ST_{c0}$-ordered space. $\square$

**Definition 3.8.** *A supra topological ordered space $(X, \mu, \preceq)$ is called $ST_{c\frac{1}{2}}$-ordered if for every $a \npreceq b$ in $X$, there exist an increasing supra open set $G$ containing $a$ such that $b$ is a supra limit point for $G^c$ or a decreasing supra open set $H$ containing $b$ such that $a$ is a supra limit point for $H^c$.*

**Example 3.9.** Consider $\mu = \{\emptyset, X, \{a\}\}$ is a supra topology and $\preceq = \triangle \bigcup \{(b, a)\}$ is a partial order relation on $X = \{a, b\}$. Then we have

$a \not\preceq b$. It can be seen that $a \in i(a)$ which is an increasing supra open set and $b \in (\{a\}^c)^{sl} = \{b\}$. Thus $(X, \mu, \preceq)$ is $ST_{c_{\frac{1}{2}}}$-ordered.

**Remark 3.10.** *The given supra topological ordered space in Example 3.2 is not $ST_{c_{\frac{1}{2}}}$-ordered space, because the elements* 1 *and* 2 *are not supra limit points for any subset of* $\mathbb{R}$.

**Proposition 3.11.** *If $(X, \mu, \preceq)$ is an $ST_{c_{\frac{1}{2}}}$-ordered space, then $X^{sl} = X$ or $X^{sl} = X \setminus \{a\}$.*

**Proof.** Suppose, to the contrary, that there exist two elements $a, b$ in $X$ such that $X^{sl} = X \setminus \{a, b\}$. From the definition of $\preceq$, we get that either $a \not\preceq b$ or $b \not\preceq a$. Say, $a \not\preceq b$. Since $(X, \mu, \preceq)$ is $ST_{c_{\frac{1}{2}}}$-ordered, then there exists an increasing supra open set $G$ containing $a$ such that $b$ is a supra limit point for $G^c$ or a decreasing supra open set $H$ containing $b$ such that $a$ is a supra limit point for $H^c$. This means that $a$ or $b$ are supra limit points of $X$. So, we obtain a contradiction with our assumption. Hence the proposition holds. □

**Corollary 3.12.** *Any $ST_{c_{\frac{1}{2}}}$-ordered space contains at most one singleton supra open set.*

The converse of the above proposition need not be true in general as shown in Example 3.3, where $X^{sl} = X \setminus \{a\}$, but $(X, \mu, \preceq)$ is not $ST_{c_{\frac{1}{2}}}$-ordered.

**Definition 3.13.** *A supra topological ordered space $(X, \mu, \preceq)$ is called $ST_{c_1}$-ordered if for every $a \not\preceq b$ in $X$, there exist an increasing supra open set $G$ containing $a$ such that $b \in G^c$ and a decreasing supra open set $H$ containing $b$ such that $a$ is a supra limit point for $H^c$.*

The proofs of the following two propositions are easy and so will be omitted.

**Proposition 3.14.** *Every minimal element of an $ST_{c_1}$-ordered space $(X, \mu, \preceq)$ is a supra limit point of $X$.*

**Proposition 3.15.** *If an $ST_{c_1}$-ordered space $(X, \mu, \preceq)$ does not contain maximal (minimal) element, then $X^{sl} = X$.*

**Example 3.16.** Consider $\nu = \{\emptyset, G \subseteq \mathbb{R}$ such that $G = (-\infty, b)$ or $(a, \infty)$ or their union $\}$ is a supra topology on $\mathbb{R}$. Let $X = (0,1)$ and $(X, \mu)$ be a subspace of $(\mathbb{R}, \nu)$. We define a supra topology on $[0,1)$ as follows: $\mu^* = \{G^* : G^* = G \text{ or } G^* = G \bigcup \{0\},\ G \in \mu\}$ and consider $\preceq$ is the usual partial order relation on $[0,1)$. Then $([0,1), \mu^\star, \preceq)$ is a supra topological ordered space. In the following, we point out that the condition of an $ST_{c_1}$-ordered space is satisfied. For each $a \in (0,1)$, we have $a \npreceq 0$. Then $a \in G^* = (\frac{1}{2}a, 1)$ which is an increasing supra open set and $0 \in (G^*)^c = [0, \frac{1}{2}a]$. Also, $0 \in H^* = \{0\}$ which is a decreasing supra open set and $a \in ((H^*)^c)^{sl} = (0,1)$.

On the other hand, for each $x \npreceq y$ such that $x \neq 0$, we choose $r > 0$ to satisfies that $x - y = 2r$. Then $x \in G^* = (x - r, 1)$ which is an increasing supra open set and $y \in (G^*)^c = [0, x - r]$. Also, $y \in H^* = [0, y + r)$ which is a decreasing supra open set and $a \in ((H^*)^c)^{sl} = [y + r, 1)$.

Hence $([0,1), \mu^\star, \preceq)$ is an $ST_{c_1}$-ordered space.

**Remark 3.17.** *The given supra topological ordered space in Example 3.9 is not an $ST_{c_1}$-ordered space, as $a \npreceq b$ and there does not exist a supra open set containing $b$ which does not contain $a$.*

**Definition 3.18.** *A supra topological ordered space $(X, \mu, \preceq)$ is called $ST_{c_1\frac{1}{2}}$-ordered if for every $a \npreceq b$ in $X$, there exist an increasing supra open set $G$ containing $a$ such that $b$ is a supra limit point for $G^c$ and a decreasing supra open set $H$ containing $b$ such that $a$ is a supra limit point for $H^c$.*

**Example 3.19.** Consider $\mu = \{\emptyset, G \subseteq \mathbb{R}$ such that $G = (-\infty, b)$ or $(a, \infty)$ or their union $\}$ is a supra topology on $\mathbb{R}$ and $\preceq$ is the usual partial order relation on $\mathbb{R}$. Consider $a \npreceq b$. We choose $r > 0$ to satisfies that $a - b = 3r$. Then we can note the following:

$a \in (a - r, \infty)$ which is increasing supra open set and $b \in (-\infty, a - r]^{sl} = (-\infty, a - r]$. Also, $b \in (-\infty, b + r)$ which is decreasing supra open set and $a \in [b + r, \infty)^{sl} = [b + r, \infty)$.

Thus $(\mathbb{R}, \mu, \preceq)$ is an $ST_{c_1\frac{1}{2}}$-ordered space.

**Remark 3.20.** *We observe in Example 3.16 that $\{0\}$ is a supra open* *subset of* $([0,1),\mu^\star,\preceq)$*. This means that* $0$ *is not a supra limit point for any subset of* $([0,1),\mu^\star,\preceq)$*. So that supra topological ordered space is not* $ST_{c_1\frac{1}{2}}$*-ordered.*

**Proposition 3.21.** *If* $(X,\mu,\preceq)$ *is an* $ST_{c_1\frac{1}{2}}$*-ordered space, then* $X^{sl}=X$*.*

**Proof.** For any $a\in X$, there exists $b\in X$ such that $a\not\preceq b$ or $b\not\preceq a$. Say, $a\not\preceq b$. By hypotheses, there exists a decreasing supra open set $H$ containing $b$ such that $a$ is a supra limit point for $H^c$. Obviously, $H^c\subseteq X$. So $a$ is a supra limit point for $X$. Thus $X\subseteq X^{sl}$. Hence $X^{sl}=X$. $\square$

**Corollary 3.22.** *There is no a singleton open subset of any* $ST_{c_1\frac{1}{2}}$*-ordered space.*

The converse of the above proposition need not be true in general as the following example illustrates.

**Example 3.23.** consider $\mu=\{\emptyset,X,\{a,b\},\{b,c\},\{a,c\}\}$ is a supra topology and $\preceq=\triangle\bigcup\{(a,c)\}$ is a partial order relation on $X=\{a,b,c\}$. Then $X^{sl}=X$. Whereas $(X,\mu,\preceq)$ is not an $ST_{c_1\frac{1}{2}}$-ordered space.

**Definition 3.24.** *A supra topological ordered space* $(X,\mu,\preceq)$ *is called* $ST_{c_2}$ *-ordered if for every* $a\not\preceq b$*, there exist disjoint an increasing supra open set* $G$ *containing* $a$ *and a decreasing supra open set* $H$ *containing* $b$ *such that* $a\in(H^c)^{sl}$*,* $b\in(G^c)^{sl}$ *and* $(H^c)^{sl}\bigcap(G^c)^{sl}=\emptyset$*.*

**Example 3.25.** Consider $\mu=\{G\subseteq\mathbb{R}$ such that $G=(-\infty,b]$ or $[a,\infty)$ or their union $\}$ is a supra topology and $\preceq$ is the usual partial order relation on $\mathbb{R}$. Let $a\not\preceq b$. Then we choose $r\in\mathbb{R}$ to satisfies that $a-b=2r$. So we can see that $a\in(a-r,\infty)$ which is an increasing supra open set and $b\in(-\infty,a-r]^{sl}=(-\infty,a-r)$. Also, we can see that $b\in(-\infty,b+r]$ which is a decreasing supra open set and $a\in[b+r,\infty)^{sl}=(b+r,\infty)$. Obviously, $(a-r,\infty)\bigcap(-\infty,b+r]=\emptyset$ and $(-\infty,a-r]^{sl}\bigcap[b+r,\infty)^{sl}=\emptyset$. Hence $(\mathbb{R},\mu,\preceq)$ is an $ST_{c_2}$-ordered space.

**Remark 3.26.** *The given supra topological ordered space* $(\mathbb{R},\mu,\preceq)$ *in Example 2.19 is not* $ST_{c_2}$ *-ordered as illustrated in the following. If we* *suppose that* $(\mathbb{R}, \mu, \preceq)$ *is an* $ST_{c_2}$*-ordered space. Then for every* $a \npreceq b$*, there exist an increasing supra open set* $G$ *containing* $a$ *and a decreasing supra open set* $H$ *containing* $b$ *such that* $G = (c, \infty)$*,* $H = (-\infty, c)$ *and* $b < c < a$*. Then* $b \in (G^c)^{sl} = (-\infty, c]^{sl} = (-\infty, c]$ *and* $a \in (H^c)^{sl} = [c, \infty)^{sl} = [c, \infty)$*. But the intersection of* $(G^c)^{sl}$ *and* $(H^c)^{sl}$ *is not empty and this is a contradiction. Hence* $(\mathbb{R}, \mu, \preceq)$ *is not an* $ST_{c_2}$*-ordered space.*

**Definition 3.27.** *A supra topological space* $(X, \mu)$ *is called:*

**(i)** *Supra connected provided that* $X$ *is not a union of two disjoint non-empty supra open (supra closed) sets.*

**(ii)** *Supra hyperconnected (resp. supra ultraconnected) provided that* $X$ *does not contain disjoint non-empty supra open (resp. non-empty supra closed) sets.*

**Proposition 3.28.** *Every* $ST_{c_2}$*-ordered space is not a supra ultraconnected.*

**Proof.** Straightforward. $\square$

**Theorem 3.29.** *Every* $ST_{c_2}$*-ordered space* $(X, \mu, \preceq)$ *is supra disconnected.*

**Proof.** Let $(X, \mu, \preceq)$ be an $ST_{c_2}$-ordered space. Then for each $a \npreceq b$ in $X$, there exist an increasing supra open set $G$ containing $a$ and a decreasing supra open set $H$ containing $b$ such that

$$H \bigcap G = \emptyset \quad (1) \quad and \quad (H^c)^{sl} \bigcap (G^c)^{sl} = \emptyset \quad (2)$$

$$By\ (1)\ we\ obtain\ \ that \quad : H^c \bigcup G^c = X \quad (3)$$

$$From\ (3)\ we\ obtain\ \ that : (H^c)^{sl} \bigcup (G^c)^{sl} \subseteq X^{sl} = X \quad (4)$$

Now, suppose that there exists $x \in X$ such that $x \notin (H^c)^{sl}$ and $x \notin (G^c)^{sl}$. Then there exist two supra open sets $U$ and $V$ such that $U \bigcap H^c \subseteq \{x\}$ and $V \bigcap G^c \subseteq \{x\}$. This implies that $U \subseteq \{x\} \bigcup H$ and $V \subseteq \{x\} \bigcup G$. It follows that $U \bigcap V \subseteq \{x\}$. So $U \bigcap V = \{x\}$. This means that $\{x\}$ is supra open. But this contradicts Proposition 3.21. Consequently, $x \in (H^c)^{sl}$ or $x \in (G^c)^{sl}$. Hence

$$(H^c)^{sl} \bigcup (G^c)^{sl} = X \quad (5)$$

From (2) and (5), we infer that $X$ is supra disconnected. $\square$

**Corollary 3.30.** *Every $ST_{c_2}$-ordered space is not a supra hyperconnected.*

**Proposition 3.31.** *Every $ST_{c_i}$-ordered space $(X, \mu, \preceq)$ is $ST_{c_{i-\frac{1}{2}}}$-ordered, for each $i = \frac{1}{2}, 1, 1\frac{1}{2}, 2$.*

**Proof.** The proof is immediately obtained from the definitions of $ST_{c_i}$-ordered spaces. $\square$

From Remark 3.10, Remark 3.17, Remark 3.20 and Remark 3.26, we see that the converse of the above proposition need not be true in general.

We clarify the relationships among $ST_{c_i}$-ordered space in the following figure.

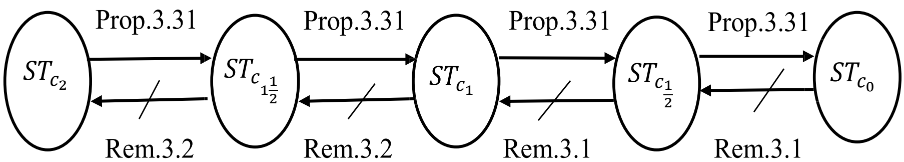


**Figure 1.** The relationships among $ST_{c_i}$-ordered space.

**Proposition 3.32.** *Every $ST_{c_i}$-ordered space $(X, \mu, \preceq)$ is an $ST_i$-space, for each $i = 0, 1, 2$.*

**Proof.** We shall start with the proof for $i = 0$, because the proofs for $i = 1$ and $i = 2$ are analogous. For all $a \neq b$ in $X$, either $a \npreceq b$ or $b \npreceq a$. Say, $a \npreceq b$. Since the space is $ST_{c_0}$-ordered, then there exist an increasing supra open set $G$ containing $a$ such that $b$ belongs to $G^c$ or a decreasing supra open set $H$ containing $b$ such that $a$ belongs to $H^c$. Thus $(X, \mu, \preceq)$ is an $ST_0$-space. $\square$

**Corollary 3.33.** *If an $ST_{c_0}$-ordered space $(X, \mu, \preceq)$, then $scl(\{a\}) \neq scl(\{b\})$, for each $a \neq b$.*

**Corollary 3.34.** *Every singleton subset of an $ST_{c_1}$-ordered space $(X, \mu, \preceq)$ is supra closed.*

**Corollary 3.35.** *If $(X, \mu, \preceq)$ is an $ST_{c_2}$-ordered space, then $\{a\} = \bigcap\{F_i :\ F_i$ is a supra closed neighborhood of $a\}$, for each $a \in X$.*

The converse of the above proposition need not be true as shown in the following:

**(i)** It can be seen, from Example 3.3, that an $ST_0$-space need not be $ST_{c_0}$-ordered.

**(ii)** In the following example, we show that an $ST_2$-space need not be $ST_{c_1}$-ordered space.

**Example 3.36.** Let $\mu = \{\emptyset, X, \{a, b\}, \{b, c\}, \{a, c\}, \{a\}, \{c\}\}$ be a supra topology and $\preceq = \triangle \bigcup\{(a, b)\}$ be a partial order relation on $X = \{a, b, c\}$. It is clearly that $(X, \tau, \preceq)$ is an $ST_1$-space, but is not an $ST_{c_2}$-ordered space.

**Theorem 3.37.** *Every $ST_{c_i}$-ordered space $(X, \mu, \preceq)$ is $SST_i$-ordered, for each $i = 1, 2$.*

**Proof.** We shall start with the proof for $i = 1$, because the proof for $i = 2$ is analogous. Suppose that $a \not\preceq b$ in $X$. By hypotheses, there exist an increasing supra open set $G$ containing $a$ such that $b$ belongs to $G^c$ and a decreasing supra open set $H$ containing $b$ such that $a$ is a supra limit point for $H^c$. Now, $a$ is a supra limit point for $H^c$ and $H^c$ is supra closed. It follows, by Proposition 2.15, that $a$ belongs to $H^c$. Hence $(X, \mu, \preceq)$ is an $SST_1$-ordered space. □

**Corollary 3.38.** *If $(X, \mu, \preceq)$ is an $ST_{c_1}$-ordered space, then $i(a)$ and $d(a)$ are supra closed sets, for each $a \in X$.*

**Corollary 3.39.** *If $a$ is the smallest (resp. largest) element in an $ST_{c_1}$-ordered space $(X, \mu, \preceq)$, then $\{a\}$ is a decreasing (resp. an increasing) supra closed.*

**Corollary 3.40.** *If $(X, \mu, \preceq)$ is an $ST_{c_2}$-ordered space, then the graph of the partially ordered set $(X, \preceq)$ is a supra closed subset of the product space $X \times X$.*

The converse of the above theorem fails as shown in Example 3.36.

**Definition 3.41.** *Consider* $(X, \mu)$ *and* $(Y, \theta)$ *are two supra topological spaces. A map* $f : (X, \mu) \to (Y, \theta)$ *is said to be:*

**(i)** $S^\star$*-open if the image of any supra open subset of* $X$ *is a supra open subset of* $Y$*.*

**(ii)** $S^\star$*-homeomorphism if it is bijective,* $S^\star$*-continuous and* $S^\star$*-open.*

**Theorem 3.42.** *If* $f : (X, \mu) \to (Y, \theta)$ *is an* $S^\star$*-homeomorphism map, then* $f(E^{sl}) = (f(E))^{sl}$*, for each* $E \subseteq X$*.*

**Proof.** Let $a \notin (f(E))^{sl}$. Then there exists $G \in \theta$ containing $a$ such that $(G \setminus \{a\}) \bigcap f(E) = \emptyset$. So $f^{-1}[(G \setminus \{a\}) \bigcap f(E)] = f^{-1}(\emptyset)$. This implies that $(f^{-1}(G) \setminus f^{-1}(a) \bigcap E = \emptyset$. Thus $f^{-1}(a) \notin E^{sl}$. Since $f$ is bijective, then $a \notin f(E^{sl})$. So $f(E^{sl}) \subseteq (f(E))^{sl}$. By reversing the preceding steps, we find that $(f(E))^{sl} \subseteq f(E^{sl})$. Hence the proof is complete. $\square$

**Theorem 3.43.** *Let* $f : (X, \mu, \preceq_1) \to (Y, \theta, \preceq_2)$ *be an ordered embedding* $S^\star$*-homeomorphism map. Then* $(X, \mu, \preceq_1)$ *is* $ST_{c_i}$*-ordered if and only if* $(Y, \theta, \preceq_2)$ *is* $ST_{c_i}$*-ordered, for each* $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$*.*

**Proof.** We prove the theorem in case of $i = 2$ and the other cases are made similarly.

*Necessity*: Suppose that $(X, \mu, \preceq)$ is an $ST_{c_2}$-ordered space and let $x \npreceq_2 y$ in $X$. Then there exist $a, b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Since $f$ is an ordered embedding map, then $a \npreceq_1 b$. Therefore there exist disjoint an increasing supra open set $G$ containing $a$ and a decreasing supra open set $H$ containing $b$ such that $a \in (H^c)^{sl}$, $b \in (G^c)^{sl}$ and $(H^c)^{sl} \bigcap (G^c)^{sl} = \emptyset$. So $x \in f(G)$ which is an increasing supra open set and $y \in f(H)$ which is a decreasing supra open set. Obviously, $f(G) \bigcap f(H) = \emptyset$. Since $f$ is an $S^\star$-homeomorphism map, then $f((E^c)^{sl}) = ((f(E))^c)^{sl}$, for any subset $E$ of $X$. Therefore $y \in f((G^c)^{sl}) = ((f(G))^c)^{sl}$, $x \in f((H^c)^{sl}) = ((f(H))^c)^{sl}$ and $((f(G))^c)^{sl} \bigcap ((f(H))^c)^{sl} = \emptyset$. Thus $(Y, \theta, \preceq_2)$ is an $ST_{c_2}$-ordered space.

*Sufficiency*: In a similar way, we prove that if $(Y, \theta, \preceq_2)$ is an $ST_{c_2}$-ordered space, then $(X, \mu, \preceq_1)$ is an $ST_{c_2}$-ordered space. $\square$

**Proposition 3.44.** *The property of being $ST_{c_0}$-ordered space is a hereditary property.*

**Proof.** Let $(A, \mu_A, \preceq \bigcap A \times A)$ be a subspace of an $ST_{c_0}$-ordered space $(X, \mu, \preceq)$. Then for each $a \npreceq b$ in $A$, there exists an increasing open set $G$ containing $a$ such that $b$ belongs to $G^c$ or a decreasing open set $H$ containing $b$ such that $a$ belongs to $H^c$. Say, there exists an increasing open set $G$ containing $a$ such that $b$ belongs to $G^c$. Now, $a \in U = G \bigcap A$ and $b \notin U$. It follows, by Proposition 2.11, that $U$ is an increasing supra open subset of $(A, \tau_A, \preceq \bigcap A \times A)$. Hence $(A, \mu_A, \preceq \bigcap A \times A)$ is an $ST_{c_0}$-ordered space. □

**Lemma 3.45.** *Consider $(A, \tau_A)$ is a subspace of $(X, \tau)$ and let $cl_A$, $int_A$ and $sl$ stand for the supra closure, supra interior and supra limit operators, respectively, in $(A, \tau_A)$. Then:*

**(i)** $cl_A(U) = cl(U) \bigcap A$ *for each* $U \subseteq A$.

**(ii)** $int(U) = int_A(U) \bigcap int(A)$ *for each* $U \subseteq A$.

**(iii)** $(U^{sl})_A = U^{sl} \bigcap A$ *for each* $U \subseteq A$.

**Theorem 3.46.** *Every m-set ordered subspace of $ST_{c_i}$-ordered space is $ST_{c_i}$ -ordered, for each $i = \frac{1}{2}, 1, 1\frac{1}{2}, 2$.*

**Proof.** We shall start with the proof for $i = 2$, because the proofs for the other cases are analogous. Let $(A, \mu_A, \preceq \bigcap A \times A)$ be an m-set ordered subspace of an $ST_{c_2}$-ordered space $(X, \mu, \preceq)$. For each $a \npreceq b$ in $A$, there exist disjoint an increasing supra open set $U$ containing $a$ and a decreasing supra open set $V$ containing $b$ such that $a \in (V^c)^{sl}$, $b \in (U^c)^{sl}$ and $(V^c)^{sl} \bigcap (U^c)^{sl} = \emptyset$. Obviously, $a \in G_A = U \bigcap A$ which is an increasing supra open subset of $(A, \tau_A, \preceq_A)$ and $b \in G^c_A = U^c \bigcap A$. Assume that $b \notin (G^c_A)^{sl} = (U^c \bigcap A)^{sl} \bigcap A$. Then $b \notin (U^c \bigcap A)^{sl}$. Therefore there exists $D \in \mu$ such that $b \in D$ and $D \bigcap U^c \bigcap A \subseteq \{b\}$. Since $A$ is m-set, then $L = A \bigcap D$ is a non-empty supra open set contains $b$, then $L \bigcap A \subseteq \{b\}$. But this contradicts that $b \in (U^c)^{sl}$. Thus $b \in (G^c_A)^{sl}$. Similarly, $b \in H_A = V \bigcap A$ which is a decreasing open subset of $(A, \mu_A, \preceq_A)$ and $a \in (H_A^c)^{sl}$. We can observe that $G_A \bigcap H_A = \emptyset$ and $(G_A^c)^{sl} \bigcap (H_A^c)^{sl} = \emptyset$. Hence, $(A, \mu_A, \preceq_A)$ is an $ST_{c_2}$-ordered space. □

**Lemma 3.47.** *Let $G_i$ and $H_j$ be subsets of $X$ for each $j \in J$ and $k \in K$. Then* $\bigcup\limits_{(j,k)\in J\times K} G_j \times H_k = (\bigcup\limits_{j\in J} G_j) \times (\bigcup\limits_{k\in K} H_j)$.

**Proof.** $\bigcup\limits_{(j,k)\in J\times K} G_j \times H_k = \{(a,b) : (a,b) \in G_j \times H_k$ for some $(j,k) \in J \times K\}$

$$= \{(a,b) : a \in G_j \text{ for some } j \in J \text{ and } b \in H_k \text{ for some } k \in K\}$$
$$= \{(a,b) : a \in \bigcup_{j\in J} G_j \text{ and } b \in \bigcup_{k\in K} H_k\}$$
$$= (\bigcup_{j\in J} G_j) \times (\bigcup_{k\in K} H_k).$$ □

**Proposition 3.48.** *If $(X, \tau)$ and $(Y, \theta)$ are supra topological spaces, then the collection $\{G_j \times H_k : G_j \in \tau, H_k \in \theta\}$ and their union forms a supra topology $\mu$ on $X \times Y$.*

**Proof.** Since $X \in \tau$, $Y \in \theta$ and $\emptyset \in \tau \bigcap \theta$, then $X \times Y \in \mu$ and $\emptyset \times \emptyset = \emptyset \in \mu$.

Let $G_j \times H_k \in \mu$, for each some $j \in J$ and $k \in K$. By hypothesis, $\bigcup\limits_{(j,k)\in J\times K} G_j \times H_k \in \mu$. Hence the proof is complete. □

**Remark 3.49.** *Henceforth, the supra topology obtained above is termed product supra topology.*

**Definition 3.50.** *Let $(\prod\limits_{j\in J} X_j, \preceq)$ be a product of partially ordered sets $(X_1, \preceq_1)$, $(X_2, \preceq_2)$, ... . Then $(a_1, a_2, ...) \preceq (b_1, b_2, ...)$ provided that $a_1 \preceq_1 b_1$, $a_2 \preceq_2 b_2$, ... .*

**Proposition 3.51.** *Let $G_j$ be a subset of a partially ordered set $(X, \preceq)$, for each $j \in J$. Then we have the following:*

**(i)** $i(\prod\limits_{j\in J} G_j) = \prod\limits_{j\in J} i(G_j)$.

**(ii)** $d(\prod\limits_{j\in J} G_j) = \prod\limits_{j\in J} d(G_j)$.

**Proof.** We only prove **(i)** and the case **(ii)** can be made similarly.

$i(\prod\limits_{j\in J} G_j) = \{(b_1, b_2, ...) : (a_1, a_2, ...) \preceq (b_1, b_2, ...)$, for some $(a_1, a_2, ...) \in \prod\limits_{j\in J} G_j\}$

$= \{(b_1, b_2, ...) : a_1 \preceq_1 b_1, a_2 \preceq_2 b_2, \ ...$ for some $a_1 \in G_1,\ a_2 \in G_2, ...\}$

$= \{(b_1, b_2, ...) : b_1 \in i(G_1),\ b_2 \in i(G_2), ...\}$

$= \prod\limits_{j\in J} i(G_j)$. □

**Corollary 3.52.** *The product of increasing (resp. decreasing) sets is always increasing (resp. decreasing).*

**Proof.** Let $G_j$ be increasing (resp. decreasing) sets, for each $j \in J$. Then the proof is obtained from the fact that $i(\prod\limits_{j\in J} G_j) = \prod\limits_{j\in J} i(G_j) = \prod\limits_{j\in J} G_j$ (resp. $d(\prod\limits_{j\in J} G_j) = \prod\limits_{j\in J} d(G_j) = \prod\limits_{j\in J} G_j$). □

Now, we are ready to prove the following significant theorem.

**Theorem 3.53.** *The finite ordered product of $ST_{c_i}$-ordered spaces is also an $ST_{c_i}$-ordered space, for all $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}$.*

**Proof.** Without loss of generality, we prove the theorem for two $ST_{c_i}$-ordered spaces and one can be made the proof for any finite number of $ST_{c_i}$-ordered spaces similarly.

Let $(X_1, \tau_1, \preceq_1)$ and $(X_2, \tau_2, \preceq_2)$ be two $ST_{c_i}$-ordered spaces and $(X_1 \times X_2, \tau, \preceq)$ be their product ordered space. Assume that $a \npreceq b$ in $X_1 \times X_2$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Then $a_1 \npreceq_1 b_1$ or $a_2 \npreceq_2 b_2$. Say $a_1 \npreceq_1 b_1$. Now we consider the following:

**(i)** If $i = 0$, then there exists an increasing supra open subset $G$ of $X_1$ containing $a_1$ such that $b_1 \in G^c$ or a decreasing supra open subset $H$ of $X_1$ containing $b_1$ such that $a_1 \in H^c$. Say, there exists an increasing supra open subset $G$ of $X_1$ containing $a_1$ such that $b_1 \in G^c$. So $G \times X_2$ is an increasing supra open set containing $a$ such that $b \in (G \times X_2)^c = G^c \times X_2$. Hence $(X_1 \times X_2, \tau, \preceq)$ is $ST_{c_0}$-ordered.

**(ii)** If $i = \frac{1}{2}$, then there exist an increasing supra open subset $G$ of $X_1$

containing $a_1$ such that $b_1 \in (G^c)^{sl}$ or a decreasing supra open subset $H$ of $X_1$ containing $b_1$ such that $a_1 \in (H^c)^{sl}$. Say, there exists an increasing supra open subset $G$ of $X_1$ containing $a_1$ such that $b_1 \in (G^c)^{sl}$. Then $G \times X_2$ is an increasing supra open set containing $a$. Suppose, to the contrary, that $b \notin (G^c \times X_2)^{sl}$. Then there exists a supra open subset $U \times V$ of $X_1 \times X_2$ containing $b$ such that $(U \times V) \bigcap (G^c \times X_2) \subseteq \{b\}$. This implies that $(U \bigcap G^c) \times (V \bigcap X_2) \subseteq \{b\}$. So $U \bigcap G^c \subseteq \{b_1\}$. But this contradicts that $b_1 \in (G^c)^{sl}$. The contradiction arises by supposing that $b \notin (G^c \times X_2)^{sl}$. Hence $(X_1 \times X_2, \tau, \preceq)$ is $ST_{c_{\frac{1}{2}}}$-ordered.

**(iii)** The proofs for $i = 1, 1\frac{1}{2}$ are similar to the proof of $i = \frac{1}{2}$ above. □

## 4. Conclusion

The notions of $ST_{c_i}$-ordered spaces ($i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$) are introduced and investigated in this study. With the help of examples, the relationships among the new ordered spaces are shown and the relationships between these ordered spaces and some $SST_i$-ordered spaces are illustrated. Some topological concepts such as $S^\star$-homeomorphism maps, supra ultraconnected spaces and supra continuous topological spaces are established and their relationships with $ST_{c_i}$-ordered spaces are studied. In addition, some results which connect some of $ST_{c_i}$-ordered spaces with some algebra notions are investigated. Two of the significant results are Theorem 3.29 which state that every $ST_{c_2}$-ordered space is supra disconnected and Theorem 3.53 which state that the finite ordered product of $ST_{c_i}$-ordered spaces is $ST_{c_i}$-ordered, for all $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}$. In an upcoming paper, we plan to utilize a notion of somewhere dense sets [3, 8] to generalized the initiated spaces in this work.

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## References


[1] M. Abo-Elhamayel and T. M. Al-shami, Supra homeomorphism in supra topological ordered spaces, *FactaUniversitatis, Series: Mathematics and Informatics*, 31 (5) (2016), 1091-1106.

[2] T. M. Al-shami, Some results related to supra topological spaces, *Journal of Advanced Studies in Topology*, 7 (4) (2016), 283-294.

[3] T. M. Al-shami, Somewhere dense sets and $ST_1$-spaces, *Punjab University Journal of Mathematics*, 49 (2) (2017), 101-111.

[4] T. M. Al-shami, Supra $\beta$-bicontinuous maps via topological ordered spaces, *Mathematical Sciences Letters*, 6 (3) (2017), 239-247.

[5] T. M. Al-shami, On some maps in supra topological ordered spaces, *Journal of New Theory*, 20 (2018), 76-92

[6] T. M. Al-shami and M. E. El-Shafei, On supra soft topological ordered spaces, *Arab Journal of Basic and Applied Sciences*, 26 (1) (2019), 433-445.

[7] T. M. Al-shami and M. E. El-Shafei, Two types of separation axioms on supra soft topological spaces, *Demonstr. Math.*, 52 (1) (2019), 147-165.

[8] T. M. Al-shami and T. Noiri, More notions and mappings via somewhere dense sets, *Afrika Matematika*, 30 (2019), 1011-1024.

[9] T. M. Al-shami and M. K. Tahat, I (D,B)-supra pre maps via supra topological ordered spaces, *Journal of Progressive Research in Mathematics*, 12 (3) (2017), 1989-2001.

[10] S. D. Arya and K. Gupta, New separation axioms in topological ordered spaces, *Indain Journal of Pure and Appllied Mathematics*, 22 (1991), 461-468.

[11] B. A. Asaad, M. K. Tahat, and T. M. Al-shami, Supra b maps via topological ordered spaces, *European Journal of Pure and Applied Mathematics*, 12 (3) (2019), 1231-1247.

[12] P. Das, Separation axioms in ordered spaces, *Soochow Journal of Mathematics*, 30 (2004), 447-454.

[13] M. E. El-Shafei, M. Abo-Elhamayel, and T. M. Al-shami, Strong separation axioms in supra topological ordered spaces, *Mathematical Sciences Letters*, 6 (3) (2017), 271-277.

[14] M. E. El-Shafei, M. Abo-Elhamayel, and T. M. Al-shami, Generating ordered maps via supra topological ordered spaces, *International Journal of Modern Mathematical Sciences*, 15 (3) (2017), 339-357.

[15] M. E. El-Shafei, M. Abo-Elhamayel, and T. M. Al-shami, Supra $R$-homeomorphism in supra topological ordered spaces, *International Journal of Algebra and Statistics*, 6 (1-2) (2017), 158-167.

[16] J. L. Kelley, *General topology*, Springer Verlag, 1975.

[17] M. K. R. S. V. Kumar, Homeomorphism in topological ordered spaces, *Acta Ciencia Indica*, XXVIII(M)(1) (2012), 67-76.

[18] D. S. Leela and G. Balasubramanian, New separation axioms in ordered topological spaces, *Indian Journal of Pure and Applied Mathematics*, 33 (2002), 1011-1016.

[19] A. S. Mashhour, A. A. Allam, F. S. Mahmoud, and F. H. Khedr, On supra topological spaces, *Indain Journal of Pure and Appllied Mathematics*, 14 (1983), 502-510.

[20] S. D. McCartan, Separation axioms for topological ordered spaces, *Proc. Camb. Phil. Soc.*, 64 (1968), 965-973.

[21] S. D. McCartan, Bicontinuous preordered topological spaces, *Pacific Journal of Mathematics*, 38 (1971), 523-529.

[22] L. Nachbin, *Topology and ordered*, D. Van Nostrand Inc. Princeton, New Jersey, (1965).

[23] R. B. Yasseen, ON m-Topological space, University of Tikrit, Tikrit, Iraq, (2008).

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