Journal of Mathematical Extension
Vol. 14, No. 4, (2020), 67-90
ISSN: 1735-8299
URL: http://www.ijmex.com
Original Research Paper

Continuity and Fixed Point of a New Extension of F-Suzuki-Contraction Mappings in b-Metric Spaces With Application

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Abstract. In this paper, firstly, we introduce a new extension of F-Suzuki-contraction mappings namely generalized F_p-Suzuki contraction. Moreover, we prove a fixed point theorem for such contraction mappings even without considering the completeness condition of space. In the following, we respond the open question of Rhoades(see Rhoades [26], p.242) regarding existence of a contractive definition which is strong enough to generate a fixed point but dose not force the mapping to be continuous at the fixed point. Also, we provide some examples show that our main theorem is a generalization of previous results. Finally, we give an application to the boundary value problem of a nonlinear fractional differential equation for our results.

AMS Subject Classification: 37C25; 26A15
Keywords and Phrases: Fixed point, continuity, SO-b-complete, F_p-suzuki-contraction mapping

Received: July 2018; Accepted: June 2019
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1. Introduction

In recent decades, a great attention has been focused on the study of fixed point theorems(see [5, 6, 10, 11]). The F-contraction is a new contraction which firstly defined by Wardowski [28] and generalization of the Banach contraction principle. This concept has been expanded to F-weak contraction by Wardowski and Dung [29]. Also, Dung and Hang [14] extended some fixed point theorems by introduce the new notation of generalized F-contraction.

Throughout this paper, we denote by $\mathfrak{F}$, the set of all functions satisfying the conditions:

(F_1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;

(F_2) $\inf F = \infty$;

(F_3) F is continuous on $(0, \infty)$.

In 2014, Piri and Kumam [23] define the concept of F-Suzuki contraction and give a new version of fixed point which generalizes the result of Wardowski as follows.

Definition 1.1. ([23]) *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-Suzuki contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$ and $\frac{1}{2}d(x, Tx) < d(x, y)$,*

$$\tau + F(d(Tx, Ty)) \leqslant F(d(x, y)),$$

where $F \in \mathfrak{F}$.

Theorem 1.2. ([23]) *Let (X, d) be a complete metric space and $T : X \to X$ be an F-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

Recently, some researchers have studied the existence of fixed point theorems in b-metric spaces, for instance see [2, 3, 8, 9, 12, 13, 16, 19, 20, 24]. As, Piri and Kumam [24] by introducing a generalized F-Suzuki contraction in b-metric spaces extended the some of previous results as follows.

Definition 1.3. ([4]) *Let X be a nonempty set and $s \geqslant 1$ be given real numbers. A mapping $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:*

(a_1) *$d(x, y) = 0$ if and only if $x = y$;*

(a_2) *$d(x, y) = d(y, x)$;*

(a_3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) *is called a b-metric space (with constant* s*).*

Definition 1.4. ([27]) *Let* (X, d) *be a b-metric space and* $\{x_n\}$ *be a sequence in* X*. We say that*

(b_1) $\{x_n\}$ *b-converges to* $x \in X$ *if* $d(x_n, x) \longrightarrow 0$ *as* $n \longrightarrow \infty$*;*

(b_2) $\{x_n\}$ *is a b-Cauchy sequence if* $d(x_m, x_n) \longrightarrow 0$ *as* $m, n \longrightarrow \infty$*;*

(b_3) (X, d) *is b-complete if every b-Cauchy sequence in* X *is b-convergent.*

Let $\mathfrak{F}_G$ be the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ such that F is strictly increasing and continuous on $(0, \infty)$ and Ψ be a collection of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Ψ_1) ψ is continuous;

(Ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.5. ([25]) *Let* (X, d) *be a b-metric space with constant* s*. A self-mapping* $T : X \to X$ *is said to be a generalized* F*-Suzuki-contraction if there exist* $F \in \mathfrak{F}_G$ *such that, for all* $x, y \in X$ *with* $x \neq y$ *and* $\frac{1}{2s} d(x, Tx) < d(x, y)$*,*

$$F(s^5 d(Tx, Ty)) \leqslant F(M_T(x, y)) - \psi(M_T(x, y)),$$

where $\psi \in \Psi$ *and*

$$\begin{aligned} M_T(x, y) = \max\Big\{ & d(x, y), d(T^2x, y), \frac{d(Tx, y) + d(x, Ty)}{2s}, \\ & \frac{d(T^2x, x) + d(T^2x, Ty)}{2s}, d(T^2x, Ty) + d(T^2x, Tx), \\ & d(T^2x, Ty) + d(Tx, x), d(Tx, y) + d(y, Ty) \Big\}. \end{aligned}$$

Theorem 1.6. ([24]) *Let* (X, d) *be a b-complete and* $T : X \to X$ *be an generalized* F*-Suzuki-contraction. Then,* T *has a unique fixed point* $x^* \in X$ *and for every* $x \in X$ *the sequence* $\{T^n x\}_{n=1}^{\infty}$ *b-converges to* x^**.*

In fact, in all of the above cases, we observe that the mapping is continuous at the fixed point. In [26], Rhoades posed an open question as to whether it is possible to define contractions which, by using them, proved the existence of a fixed point, which does not have to be continuous in that mapping. Some authors such as Kannan [17, 18], Pant [21], Bisht and Pant [7], A. Panta and R. P. Panta [22] and etc provide solutions to the open problem on the existence of a contraction mapping which possesses a fixed point but not continuous at the fixed point. In this paper, we address the following questions.

(Q_1) Is it possible to remove the completeness assumption of the space in Theorem 1.6. ?

(Q_2) Is it possible to consider a more extension contraction than generalized F-Suzuki-contraction in Theorem 1.6?

(Q_3) Is condition generalized F-Suzuki-contraction in Theorem 1.6 have to satisfy all the x and y, or not, we can limit it?

(Q_4) What about the continuity of the function T at its fixed point?

In future, we show that Theorem 1.6 is hold whenever X is not a complete b-metric space. For this purpose, we applying the notation of the orthogonal sets that was first introduced by the Eshaghi Gordgi et al. [15] as follows.

Definition 1.7. ([15]) *Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If " $\perp$ " satisfies the following condition:*

$$\exists x_0\colon (\forall y, y \perp x_0) \ \ or \ \ (\forall y, x_0 \perp y),$$

then " $\perp$ " is called an orthogonality relation and the pair $(X, \perp)$ an orthogonal set(briefly O-set).

Note that in above definition, we say that x_0 is an orthogonal element. Also, we say that elements $x, y \in X$ are $\perp$-comparable either $x \perp y$ or $y \perp x$.

Definition 1.8. ([1, 25]) *Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called a strongly orthogonal sequence(briefly, SO-sequence) if*

$$(\forall n, k; \quad x_n \perp x_{n+k}) \quad or \quad (\forall n, k; \quad x_{n+k} \perp x_n).$$

Definition 1.9. ([1, 25]) *Let $(X, \perp)$ be an O-set. A mapping $T : X \to X$ is said to be $\perp$-preserving if $x \perp y$ implies $T(x) \perp T(y)$.*

Furthermore, we introduce a new contractive definition which is a generalization of generalized F-Suzuki-contraction. Also, we show that this contractive is sufficient to satisfy more limited number x and y in X to fined the fixed point. In addition, we provide a new answer to the open question posed in [26], that is, existence of contractive mappings which are discontinuous at their fixed points. In the following, we present some examples to illustrate the main results. Finally, in the last section, as an application, we consider the existence and uniqueness of a solution for a boundary value problem of a nonlinear fractional differential equation in b-metric space. Here, before the main result, we explain the following definitions.

Definition 1.10. *Let $(X, \perp, d)$ be an orthogonal b-metric space. A SO-sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that* $\lim_{n\to\infty} d(x_n, x) = 0$. *In this case, we write* $\lim_{n\to\infty} x_n = x$.

Definition 1.11. *Let $(X, \perp, d)$ be an orthogonal b-metric space. Then X is said to be $\perp$-regular if for each SO-sequence $\{x_n\}$ with $x_n \to x$ for some $x \in X$, we conclude that*

$$(\forall n; x_n \perp x) \;\; or \;\; (\forall n; x \perp x_n).$$

Definition 1.12. *Let $(X, \perp, d)$ be an orthogonal b-metric space. A mapping $f : X \to X$ is strongly orthogonal b-continuous(briefly, SO-b-continuous) in $x \in X$ if for each SO-sequence $\{x_n\}$ in X if $x_n \to x$, then $f(x_n) \to f(x)$. Also, f is SO-b-continuous on X if f is SO-b-continuous in each $x \in X$.*

Definition 1.13. *Let $(X, \perp, d)$ be an orthogonal b-metric space. X is said to be strongly orthogonal b-complete(briefly, SO-b-complete) if every b-Cauchy SO-sequence is b-convergent.*

Remark 1.14. *It is obvious that every b-complete is a SO-b-complete.*

The following examples show that the converse of Remark 1.14 is not true in general.

Example 1.15. Let $X = (0, 1]$ and $D : X \times X \to \mathbb{R}^+$ defined by $D(x, y) = (|x - y|)^2$. Thus, (X, D) is a b-metric space with $s = 2$. Define orthogonal relation " $\perp$ " as follows

$$x \perp y \quad \Leftrightarrow \quad x.y \in \{x, y\}.$$

Clearly, X is O-set with $x_0 = 1$. Obviously, X is SO-b-complete. But X is not b-complete. Because the b-Cauchy sequence $x_n = \dfrac{1}{n}$ is not b-convergent in X.

Example 1.16. Let $X = \mathbb{R}$ and $\sigma : X \times X \to \mathbb{R}^+$ define by

$$\sigma(x, y) = \begin{cases} 0, & x = y \\ \left|\left(\dfrac{n+1}{n} - \dfrac{m+1}{m}\right)\right|, & (x, y) = \left(\dfrac{n+1}{n}, \dfrac{m+1}{m}\right) \\ 4, & (x, y) \in \left\{\left(\dfrac{n+1}{n}, 0\right), \left(0, \dfrac{n+1}{n}\right)\right\} \\ 1, & \text{otherwise.} \end{cases}$$

Thus, (X, D) is a b-metric space with $s = 2$. Define orthogonal relation " $\perp$ " as follows

$$x \perp y \quad \Leftrightarrow \quad x = 0 \;\text{ or }\; 0 \leqslant x \leqslant y < 1.$$

Clearly, X is O-set with $x_0 = 0$. Obviously, X is SO-b-complete. But X is not b-complete. Because the b-Cauchy sequence $x_n = \dfrac{n+1}{n}$ is not convergent in X.

2. Main Results

In this section, we formulate our main results. We begin with the following definition.

Definition 2.1. *Let (X,d) be a b-metric space with constant s. A self-mapping $T:X\to X$ is said to be a generalized F_p-Suzuki-contraction if there exist $F\in\mathfrak{F}_G$ and $p\in\mathbb{N}$ such that, for all $x,y\in X$ with $x\neq y$ and $\frac{1}{2s}d(x,Tx)<d(x,y)$,*

$$F(p^2s^5d(T^px,T^py))\leqslant F(M_T(x,y))-\psi(M_T(x,y)), \tag{1}$$

where $\psi\in\Psi$ and $M_T(x,y)$ is the same as in Definition 1.5.

Theorem 2.2. *Let $(X,d,\perp)$ be a SO-b-complete(not necessarily b-complete) with orthogonal element x_0. Let $T:X\to X$ be an generalized $FP_\perp$-Suzuki-contraction, SO-b-continuous and $\perp$-preserving. Also, let X be $\perp$-regular. Then T has a unique fixed point $x^*\in X$ and for all $x\in X$ the sequence $\{T^nx\}$ b-converges to x^*.*

Proof. We consider the sequence $\{x_n\}$ defined by $x_n=T^nx_0,\ n=0,1,2,\ldots$. From the definition of orthogonal element x_0, we have

$$(\forall n\in\mathbb{N},\ \ x_0\perp T^nx_0=x_n)\ \text{ or }\ (\forall n\in\mathbb{N},\ \ x_n=T^nx_0\perp x_0).$$

Also, since T is $\perp$-preserving, we have

$$(\forall\, n,k\in\mathbb{N},\ \ x_n=T^nx_0\perp T^{n+k}x_0=x_{n+k})$$

or

$$(\forall\, n,k\in\mathbb{N},\ \ x_{n+k}=T^{n+k}x_0\perp T^nx_0=x_n).$$

Therefore x_n is a SO-sequence.

We have the following results:

(1) If there exists n_0 such that $d(x_{n_0},Tx_{n_0})=0$, then we have $Tx_{n_0}=x_{n_0}$, and the proof is finished.

(2) If for all n, $d(x_n,Tx_n)>0$, since $\{x_n\}$ is SO-sequence and

$$\frac{1}{2s}d(x_n,Tx_n)<d(x_n,Tx_n)=d(x_n,x_{n+1}),$$

so by the assumption of the theorem, we have

$$F(p^2s^5d(T^px_n,T^px_{n+1}))\leqslant F(M_T(x_n,x_{n+1}))-\psi(M_T(x_n,x_{n+1})). \tag{2}$$

Since

$$
\begin{aligned}
&\max\{d(x_n,x_{n+1}),d(T^2x_n,x_{n+1})\}\\
&\leqslant M_T(x_n,x_{n+1})\\
&=\max\Big\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1}),\frac{d(x_n,x_{n+2})}{2s},\frac{d(x_{n+2},x_n)}{2s},\\
&\qquad d(x_{n+2},x_{n+1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\Big\}\\
&\leqslant \max\Big\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1}),\frac{s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]}{2s},\\
&\frac{s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]}{2s},d(x_{n+2},x_{n+1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\Big\}\\
&\leqslant \max\{d(x_n,x_{n+1}),d(T^2x_n,x_{n+1})\},
\end{aligned}
$$

therefore

$$
\begin{aligned}
F(p^2s^5d(x_{n+p},x_{n+p+1}))\leqslant &F(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\})\\
&-\psi(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}).
\end{aligned}
\tag{3}
$$

We consider two cases as follows:

Case 1. Let $p=1$. In this case, if $\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}=d(x_{n+1},x_{n+2})$, so from (3), we get

$$F(d(x_{n+1},x_{n+2}))\leqslant F(d(x_{n+1},x_{n+2}))-\psi(d(x_{n+1},x_{n+2})),$$

which is a contradiction, and so we conclude that

$$F(d(x_{n+1},x_{n+2}))\leqslant F(d(x_n,x_{n+1})-\psi(d(x_n,x_{n+1}))\leqslant F(d(x_n,x_{n+1})). \tag{4}$$

Applying (4) and F_1, we have

$$d(x_{n+1},x_{n+2})\leqslant d(x_n,x_{n+1}). \tag{5}$$

Therefore $\{d(x_n,x_{n+1})\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence of real numbers, and so there exists $\delta\geqslant 0$ such that $\lim_{n\to\infty}d(x_{n+1},x_n)=\delta$. Letting $n\to\infty$ in (4), we have $F(\delta)\leqslant F(\delta)-\psi(\delta)$. This implies that $\psi(\delta)=0$ and thus $\delta=0$, that is

$$\lim_{n\to\infty}d(x_n,Tx_n)=\lim_{n\to\infty}d(x_n,x_{n+1})=0. \tag{6}$$

Case 2. Let $p>1$. We adopt the following notations:

(d_1) Let $x=x_n$ and $y=x_{n+1}$;

(d_2) $Q_n(x,y) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \cdots, d(x_{n+p-1}, x_{n+p})\}$.

Here, we break the argument into two steps, each of which illustrates something more.

Step 1: The sequence $\{Q_n(x,y)\}$ is decreasing.
For this purpose, applying (3) and (F_1), we have

$$\begin{aligned} d(x_{n+p}, x_{n+p+1}) &\leqslant \frac{1}{p^2 s^5} \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \\ &\leqslant \frac{1}{p^2 s^5} Q_n(x,y) \leqslant Q_n(x,y), \end{aligned}$$

so, $Q_{n+1}(x,y) \leqslant Q_n(x,y)$ for every $n \in \mathbb{N}$.

Step 2: $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
For every $i \in \{0, 1, 2, \cdots, p-1\}$ and $n \in \mathbb{N}$, taking $x = x_{n+i}$ and $y = x_{n+i+1}$ into the inequality (3), applying Step 1, then we get that

$$\begin{aligned} d(x_{n+p+i}, x_{n+p+i+1}) &\leqslant \frac{1}{p^2 s^5} \max\{d(x_{n+i}, x_{n+i+1}), d(x_{n+i+1}, x_{n+i+2})\} \\ &\leqslant \frac{1}{p^2 s^5} Q_{n+i}(x,y) \leqslant \frac{1}{p^2 s^5} Q_n(x,y), \end{aligned}$$

so, $Q_{n+p}(x,y) \leqslant \dfrac{1}{ps^5} Q_n(x,y)$. By induction procedure, we obtain that

$$Q_{n+kp}(x,y) \leqslant \left(\frac{1}{p^2 s^5}\right)^k Q_n(x,y) \text{ for all } n, k \in \mathbb{N}.$$

Therefore $\lim_{k\to\infty} Q_{n+kp}(x,y) = 0$, for all $n \in \mathbb{N}$. Applying Step 1, we have $\lim_{n\to\infty} Q_n(x,y) = 0$ and since $d(x_n, x_{n+1}) \leqslant Q_n(x,y)$ for all $n \in \mathbb{N}$, we conclude that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

We shall prove that $\{x_n\}$ is a b-Cauchy SO-sequence. Suppose that $\{x_n\}$ is not a b-Cauchy SO-sequence. Then, there exists some $\varepsilon > 0$ and two sequences of positive integers $\{p(n)\}$ and $\{q(n)\}$ such that, for all positive integers n, we have

$$p(n) > q(n) > n, \quad d(x_{p(n)}, x_{q(n)}) \geqslant \varepsilon, \quad d(x_{p(n)-1}, x_{q(n)}) < \varepsilon. \tag{7}$$

Applying triangular inequality and (7), we have

$$\begin{aligned} \varepsilon &\leqslant d(x_{p(n)}, x_{q(n)}) \leqslant s[d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)})] \\ &\leqslant s d(x_{p(n)}, x_{p(n)-1}) + s\varepsilon, \end{aligned}$$

using (6), we get

$$\varepsilon \leqslant \limsup_{n\to\infty} d(x_{p(n)}, x_{q(n)}) \leqslant s\varepsilon. \tag{8}$$

Also, we have

$$\varepsilon \leqslant d(x_{p(n)}, x_{q(n)}) \leqslant s[d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)})], \tag{9}$$

and

$$d(x_{p(n)}, x_{q(n)+1}) \leqslant s[d(x_{p(n)}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1})]. \tag{10}$$

Then by (6), (8), (9) and (10), we can write

$$\frac{\varepsilon}{s} \leqslant \limsup_{n\to\infty} d(x_{p(n)}, x_{q(n)+1}) \leqslant s^2\varepsilon. \tag{11}$$

Similarly,

$$\frac{\varepsilon}{s} \leqslant \limsup_{n\to\infty} d(x_{q(n)}, x_{p(n)+1}) \leqslant s^2\varepsilon. \tag{12}$$

Applying (11) and triangle inequality, we have

$$d(x_{p(n)}, x_{q(n)+1}) \leqslant s[d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1})],$$

we implies that

$$\frac{\varepsilon}{s^2} \leqslant \limsup_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}). \tag{13}$$

Using (8) and the inequality

$$\begin{aligned} d(x_{p(n)+1}, x_{q(n)+1}) \leqslant & s[d(x_{p(n)+1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1})] \\ \leqslant & s^2[d(x_{p(n)+1}, x_{p(n)}) + d(x_{p(n)}, x_{q(n)})] \\ & + sd(x_{q(n)}, x_{q(n)+1}), \end{aligned}$$

we deduce that

$$\limsup_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}) \leqslant s^3\varepsilon. \tag{14}$$

Therefore from (13) and (14), we have

$$\frac{\varepsilon}{s^2} \leqslant \limsup_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}) \leqslant s^3\varepsilon. \tag{15}$$

Also, applying (8), we can show that

$$\frac{\varepsilon}{s^2} \leqslant \limsup_{n\to\infty} d(x_{p(n)+p}, x_{q(n)+p}) \leqslant s^3\varepsilon.$$

Since $\{x_n\}$ is SO-sequence, applying (6) and (8), there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$, we deduce that

$$\frac{1}{2s} d(x_{p(n)}, Tx_{p(n)}) < \frac{1}{2s}\varepsilon < d(x_{p(n)}, x_{q(n)}).$$

Therefore by assumption of theorem for all $n \geqslant n_1$, imply that

$$F(d(x_{p(n)+p}, x_{q(n)+p})) \leqslant F(M_T(x_{p(n)}, x_{q(n)})) - \psi(M_T(x_{p(n)}, x_{q(n)})). \tag{16}$$

Since

$$\begin{aligned} d(x_{p(n)}, x_{q(n)}) \leqslant\ & M_T(x_{p(n)}, x_{q(n)}) \\ &= \max\Big\{ d(x_{p(n)}, x_{q(n)}), d(x_{p(n)+2}, x_{q(n)}), \\ &\qquad \frac{d(x_{p(n)+1}, x_{q(n)}) + d(x_{p(n)}, x_{q(n)+1})}{2s}, \\ &\qquad \frac{d(x_{p(n)+2}, x_{p(n)}) + d(x_{p(n)+2}, x_{q(n)+1})}{2s}, \\ &\qquad d(x_{p(n)+2}, x_{q(n)+1}) + d(x_{p(n)+2}, x_{p(n)+1}), \\ &\qquad d(x_{p(n)+2}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{p(n)}), d(x_{p(n)+1}, \\ &\qquad x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1})\Big\} \end{aligned}$$

$$\begin{aligned} \leqslant \max\Big\{ & d(x_{p(n)}, x_{q(n)}), s[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)})], \\ & \frac{d(x_{p(n)+1}, x_{q(n)}) + d(x_{p(n)}, x_{q(n)+1})}{2s}, \\ & \frac{s[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{p(n)})]}{2s} \\ & + \frac{s[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1})]}{2s}, \\ & s[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1})] + d(x_{p(n)+2}, x_{p(n)+1}), \\ & s[d(x_{p(n)+2}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1})] d(x_{p(n)+1}, x_{p(n)}), \\ & d(x_{p(n)+1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1})\Big\}, \end{aligned}$$

let $n \to \infty$ on above inequality and using (8), (11), (12) and (15), we have

$$\varepsilon \leqslant \limsup_{n\to\infty} M_T(x_{p(n)}, x_{q(n)}) \leqslant s^3\varepsilon. \tag{17}$$

Similarly, we can see that

$$\varepsilon \leqslant \liminf_{n\to\infty} M_T(x_{p(n)}, x_{q(n)}) \leqslant s^3\varepsilon. \tag{18}$$

Letting $n \to \infty$ in (16), and applying (17) and (18), we deduce that

$$\begin{aligned}
&F(p^2 s^5 \limsup d(x_{p(n)+p}, x_{p(n)+p})) \\
\leqslant &F(\limsup_{n\to\infty} M_T(x_{p(n)}, x_{q(n)})) - \psi(\limsup_{n\to\infty} M_T(x_{p(n)}, x_{q(n)})) \\
\leqslant &F(s^3\varepsilon) - \psi(\varepsilon).
\end{aligned}$$

Since F is increasing, we have

$$\limsup_{n\to\infty} d(x_{p(n)+p}, x_{p(n)+p}) < \frac{s^3\varepsilon}{p^2 s^5} = \frac{\varepsilon}{p^2 s^2} < \frac{\varepsilon}{s^2},$$

which is a contradiction and so $\varepsilon = 0$. Therefore $\{x_n\}$ is a b-Cauchy SO-sequence. Since $(X, \perp, d)$ is SO-b-complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{19}$$

On the other hand, T is SO-b-continuous function, then

$$T(x^*) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} (x_{n+1}) = x^*,$$

that is x^* is a fixed point of T.
Now, we show that x^* is unique. For this purpose, let $y^* \in X$ be another fixed point of T. Since x_0 is an orthogonal element, by the definition of orthogonality, we have

$$(x_0 \perp y^*) \text{ or } (y^* \perp x_0).$$

Since T is $\perp$-preserving, then

$$(x_n = T^n x_0 \perp T^n y^* = y^*) \text{ or } (y^* = T^n y^* \perp T^n x_0 = x_n).$$

Therefore, y^* and x_n are comparable. Also, under assumption (2), we have for all $n \in \mathbb{N}$, $d(x_n, Tx_n) > 0$. Therefore for all $n \in \mathbb{N}$, we conclude that $d(y^*, x_n) > 0$. Since if there exists $m \in \mathbb{N}$ such that $d(y^*, x_m) = 0$, then $y^* = x_m$, and so $Tx_m = Ty^* = y^*$, that is $d(x_m, Tx_m) = 0$. Then, we have $\frac{1}{2s} d(y^*, Ty^*) \leqslant d(y^*, x_n)$ and from the assumption of the theorem, we implies that

$$F(d(y^*, x_{n+p})) = F(d(T^p y^*, T^p x_n)) \leqslant F(M_T(y^*, x_n)) - \psi(M_T(y^*, x_n)). \tag{20}$$

Since

$$\begin{aligned}
M_T(y^*, x_n) &= \max\Big\{d(y^*, x_n), d(T^2y^*, x_n), \frac{d(Ty^*, x_n) + d(y^*, Tx_n)}{2s},\\
&\qquad d(T^2y^*, Tx_n) + d(Ty^*, y^*), d(T^2y^*, Tx_n) + d(Ty^*, y^*),\\
&\qquad d(Ty^*, y^*) + d(Tx_n, x_n)\Big\}\\
&\leqslant \max\Big\{d(y^*, x_n), \frac{d(y^*, x_n) + d(y^*, x_{n+1})}{2s},\\
&\qquad \frac{d(y^*, x_{n+1})}{2s}, d(y^*, x_{n+1}), d(x_n, x_{n+1})\Big\}\\
&= \max\{d(y^*, x_n), d(y^*, x_{n+1}), d(x_n, x_{n+1})\}.
\end{aligned}$$

Letting $n \to \infty$ and using (19), we have $\lim_{n\to\infty} M_T(y^*, x_n) = d(y^*, x^*)$. Applying (20) and continuity of F and ψ, we get

$$F(d(y^*, x^*)) \leqslant F(d(y^*, x^*)) - \psi(d(y^*, x^*)).$$

This is contradiction, and so $x^* = y^*$. Finally, we proof that T is a Picard operator. Let $x \in X$ be arbitrary. We consider two cases:

Case 1. If $\frac{1}{2s} d(T^n x_0, T^{n+1}x_0) \geqslant d(T^n x_0, T^n x)$, letting $n \to \infty$ and using (19), we have $\lim_{n\longrightarrow\infty} T^{n+1}x = x^*$, and proof is finished.

Case 2. If $\frac{1}{2s} d(T^n x_0, T^{n+1}x_0) < d(T^n x_0, T^n x)$, hence for all $n \in \mathbb{N}$, we get

$$F(p^2 s^5 d(T^{n+p}x_0, T^{n+p}x)) \leqslant F(M_T(T^n x_0, T^n x)) - \psi(M_T(T^n x_0, T^n x)).$$

Letting $n \longrightarrow \infty$, we conclude that

$$\begin{aligned}
\lim_{n\longrightarrow\infty} F(p^2 s^5 d(T^{n+p}x_0, T^{n+p}x)) \leqslant{}& \lim_{n\longrightarrow\infty} F(M_T(T^n x_0, T^n x))\\
&- \lim_{n\longrightarrow\infty} \psi(M_T(T^n x_0, T^n x)).
\end{aligned}$$

Applying (F_1), (F_3) and continuity of ψ, we deduce that

$$\begin{aligned}
\lim_{n\longrightarrow\infty} d(T^{n+p}x_0, T^{n+p}x) &< \frac{1}{ps^5}\lim_{n\longrightarrow\infty} M_T(T^n x_0, T^n x)\\
&= \frac{1}{ps^5}\lim_{n\longrightarrow\infty} \max\Big\{d(T^n x_0, T^n x), d(T^{n+2}x_0, T^n x),\\
&\qquad \frac{d(T^{n+1}x_0, T^n x) + d(T^n x_0, T^{n+1}x)}{2s},\\
&\qquad \frac{d(T^{n+2}x_0, T^n x_0) + d(T^{n+2}x_0, T^{n+1}x)}{2s},\\
&\qquad d(T^{n+2}x_0, T^{n+1}x) + d(T^{n+2}x_0, T^{n+1}x_0),\\
&\qquad d(T^{n+2}x_0, T^{n+1}x) + d(T^{n+1}x_0, T^n x_0),\\
&\qquad d(T^{n+1}x_0, T^n x) + d(T^n x, T^{n+1}x)\Big\}.
\end{aligned}$$

If $p = 1$, as proved in the proof of Theorem 1.6, $\lim_{n\longrightarrow\infty} d(T^n x, T^{n+1}x) = 0$. Recalling (19), we observe

$$\begin{aligned}
d(x^*, \lim_{n\longrightarrow\infty} T^n x) &= d(x^*, \lim_{n\longrightarrow\infty} T^{n+p}x)\\
&< \frac{1}{s^5}[(x^*, \lim_{n\longrightarrow\infty} T^n x) + \lim_{n\longrightarrow\infty} d(T^n x, T^{n+1}x)]\\
&= \frac{1}{s^5}[(x^*, \lim_{n\longrightarrow\infty} T^n x).
\end{aligned}$$

This implies that $\lim_{n\longrightarrow\infty} T^n x = x^*$.
If $p > 1$, using (19) and triangle inequality, we obtain that

$$\begin{aligned}
d(x^*, \lim_{n\longrightarrow\infty} T^n x) &= d(x^*, \lim_{n\longrightarrow\infty} T^{n+p}x)\\
&< \frac{1}{p^2 s^5}[(x^*, \lim_{n\longrightarrow\infty} T^n x) + d(\lim_{n\longrightarrow\infty} T^n x, \lim_{n\longrightarrow\infty} T^{n+1}x)]\\
&\leqslant \frac{1}{p^2 s^5}[(d(x^*, \lim_{n\longrightarrow\infty} T^n x)\\
&+ s[d(\lim_{n\longrightarrow\infty} T^n x, x^*) + d(x^*, \lim_{n\longrightarrow\infty} T^{n+1}x)]\\
&\leqslant \frac{3}{p^2 s^4} d(x^*, \lim_{n\longrightarrow\infty} T^n x).
\end{aligned}$$

Hence

$$(1 - \frac{3}{p^2 s^4})d(x^*, \lim_{n\longrightarrow\infty} T^n x) \leqslant 0,$$

since $p \in \mathbb{N}$ and $p > 1$, we get $\lim_{n\longrightarrow\infty} T^n x = x^*$. This completes the proof. □

Remark 2.3. *In Theorem 2.2, we looked for fixed point of functional T in the case that the contraction* (1) *satisfied for all $\perp$-comparable elements $x, y \in X$ with $x \neq y$ and $\frac{1}{2s}d(x, Tx) < d(x, y)$, that is enough to be in a more limited number x and y. We note that, since in the entire proof process of Theorem 2.2, we use the SO-sequence $\{x_n\}$, so the above is true*

Remark 2.4. *Let $p = 1$ in Theorem 2.2, we can proof this theorem without assuming the SO-b-continuity of the T. It is easy to see that for SO-sequence $\{x_n\}$, we have $d(x_{n+1}, x_{n+2}) \leqslant d(x_n, x_{n+1})$ and $\lim_{n\to\infty} d(x_n, x^*) = 0$ by repeating the firstly steps in the proof of Theorem* 2.2. *We only show that x^* is unique fixed point of T. For this purpose, we show that for all $n \in \mathbb{N}$,*

$$\frac{1}{2s}d(x_n, Tx_n) < d(x_n, x^*) \text{ or } \frac{1}{2s}d(Tx_n, T^2x_n) < d(Tx_n, x^*). \tag{21}$$

Let by contrary, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2s}d(x_m, Tx_m) \geqslant d(x_m, x^*) \text{ and } \frac{1}{2s}d(Tx_m, T^2x_m) \geqslant d(Tx_m, x^*). \tag{22}$$

Hence

$$(2s)d(x_m, x^*) \leqslant d(x_m, Tx_m) \leqslant s[d(x_m, x^*) + d(x^*, Tx_m)],$$

which implies that

$$d(x_m, x^*) \leqslant d(x^*, Tx_m). \tag{23}$$

Applying (5) *and* (23), *we have*

$$d(Tx_m, T^2 m) < d(x_m, Tx_m) \leqslant s[d(x_m, x^*) + sd(x^*, Tx_m)] \leqslant (2s)d(Tx_m, x^*).$$

This is a contradiction, and so (21) *holds. Since X is $\perp$-regular, then x_n and x^* are $\perp$-comparable. Let $\frac{1}{2s}d(x_n, Tx_n) < d(x_n, x^*)$, under the assumption of theorem, we have*

$$F(d(x_{n+1}, Tx^*)) = F(d(Tx_n, Tx^*)) \leqslant F(M_T(x_n, x^*)) - \psi(M_T(x_n, x^*)). \tag{24}$$

Since

$$\begin{aligned} d(x^*, Tx^*) &\leqslant M_T d(x_n, x^*) \\ &= \max\Big\{d(x_n, x^*), d(x_{n+2}, x^*), \frac{d(x_{n+1}, x^*) + d(x_n, Tx^*)}{2s} \\ &\qquad \frac{d(x_{n+2}, x_n) + d(x_{n+2}, Tx^*)}{2s}, d(x_{n+2}, Tx^*) + d(x_{n+2}, x_{n+1}) \\ &\qquad d(x_{n+1}, x^*) + d(x^*, Tx^*)\Big\} \end{aligned}$$

$$\leqslant \max\Big\{d(x_n,x^*), d(x_{n+2},x^*), \frac{d(x_{n+1},x^*)+d(x_n,Tx^*)}{2s}$$
$$\frac{s[d(x_{n+2},x_{n+1})+d(x_{n+1},x_n)]+d(x_{n+2},Tx^*)}{2s},$$
$$d(x_{n+2},Tx^*)+d(x_{n+2},x_{n+1})$$
$$d(x_{n+2},Tx^*)+d(x_{n+1},x_n), d(x_{n+1},x^*)+d(x^*,Tx^*)\Big\}.$$

Applying (2), *we get*

$$\lim_{n\to\infty} M_T(x_n,x^*) = d(x^*,Tx^*).$$

The continuity of ψ *and* F, *and* (24) *imply that*

$$F(d(x^*,Tx^*) \leqslant F(M_T(x^*,Tx^*)) - \psi(M_T(x^*,Tx^*)),$$

that is $x^* = Tx^*$. *On the other hand, if let* $\frac{1}{2s}d(Tx_n,T^2x_n) < d(Tx_n,x^*)$. *Like the above process, we get* $x^* = Tx^*$. *The uniqueness of* x^* *is obtained as Theorem 2.2 and this completes the proof.*

In the following, we present the example that clearly shows the existence of a contraction mapping which possesses a fixed point without being continuous at this point, and this is exactly the answer to the open problem posed by Rhoades in [26].

Example 2.5. Let $X = (-3,3)$ and define a metric "d" on X by

$$d(x,y) = \begin{cases} 0, & x = y \\ |y|, & x = 0 \text{ and } y \neq 0 \\ \frac{3}{2}|x-y|, & \text{otherwise.} \end{cases}$$

Then (X,d) is a b-metric space with coefficient $s = \frac{3}{2}$. But it is not a metric space since the triangle inequality is not satisfied. Suppose that

$$x \perp y \iff x = 0.$$

Then $(X,\perp)$ is an O-set with orthogonal element $x_0 = 0$. Clearly, $(X,\perp,d)$ is not a b-complete, but it is SO-b-complete(In fact, if $\{x_n\}$ is an arbitrary b-Cauchy SO-sequence in X, then $x_n = 0$ for all $n \in \mathbb{N}$ and $x_n = 0$ is b-convergent to zero.). We see that X is $\perp$-regular. Let $T : X \to X$ be the mapping defined by

$$Tx = \begin{cases} \frac{x}{4}, & x \leqslant 0 \\ 2x+1, & 0 < x < 1 \\ -\frac{x}{32}, & x \geqslant 1. \end{cases}$$

Clearly T is $\perp$-preserving, since for all $x, y \in X$ such that $x \perp y$, if $x = 0$, then $Tx = 0$.
For each $\perp$-comparable elements $x, y \in X$, observed that

$$\frac{1}{2s} d(x, Tx) < d(x, y) \quad \Leftrightarrow \quad y \neq 0.$$

Now, we consider the following cases:

(A_1) If $x = 0$ and $y < 0$, then
$Tx = 0$, $T^2x = 0$, $Ty = \frac{y}{4}$ and $T^2y = \frac{y}{16}$. Therefore

$$d(T^2x, T^2y) = d(0, \frac{y}{16}) = \frac{1}{16}|y|,$$

$$M_T(x, y) = d(Tx, y) + d(y, Ty) = d(0, y) + d(y, \frac{y}{4}) = \frac{17}{8}|y|.$$

(A_2) If $x = 0$ and $0 < y < 1$, then
$Tx = 0$, $T^2x = 0$, $Ty = 2y + 1$ and $T^2y = -\frac{(2y+1)}{32}$. Therefore

$$d(T^2x, T^2y) = d(0, -\frac{(2y+1)}{32}) = \frac{1}{32}|2y + 1|,$$

$$M_T(x, y) = d(Tx, y) + d(y, Ty) = d(0, y) + d(y, 2y + 1) = \frac{5}{2}|y| + \frac{3}{2}.$$

(A_3) If $x = 0$ and $y \geqslant 1$, then
$Tx = 0$, $T^2x = 0$, $Ty = -\frac{y}{32}$ and $T^2y = \frac{y}{128}$. Therefore

$$d(T^2x, T^2y) = d(0, \frac{(y)}{128}) = \frac{1}{128}|y|,$$

$$M_T(x, y) = d(Tx, y) + d(y, Ty) = d(0, y) + d(y, -\frac{y}{32}) = \frac{163}{64}|y|.$$

In all above cases, taking $F(t) = \ln(t)$ and $\psi(t) = \frac{4}{85}t$, we have

$$F(4s^5 d(T^2x, T^2y)) \leqslant F(M_T(x, y)) - \psi(M_T(x, y)).$$

Hence, applying Theorem 2.2 and Remark 2.3, T has a unique fixed point $x = 0$. Also, it can be easily see that T is discontinuous at the fixed point $x = 0$ and Theorem 1.6 is not applicable this example.

Now, we illustrate our main results by another example.

Example 2.6. Let $X = Q$. Suppose that $x \perp y \Leftrightarrow xy \in \{x, y\}$. Clearly, X with the b-metric given by

$$\rho(x, y) = sup_{t\in[0,1]}|x(t) - y(t)|^{1.1},$$

is a b-metric space with coefficient $s = 2^{0.1}$. Furthermore $(X, \perp)$ is an O-set with orthogonal element $x_0 = 1$. Clearly, X is not a b-complete, but it is SO-b-complete(because if $\{x_n\}$ is an arbitrary b-Cauchy SO-sequence in X, then there exists SO-subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which $x_{n_k} = 1$ for each $k \geqslant 0$, and so $\{x_{n_k}\} \to 1$).
We see that X is $\perp$-regular. Let $T : X \to X$ be the mapping defined by

$$T(x) = \begin{cases} 2, & x = -6 \\ 1, & o.w.. \end{cases}$$

For all $x, y \in X$ such that $x \perp y$, if $x = 1$, then $Tx = 1$, and so $Tx \perp Ty$. Similarly, if $y = 1$, we have $Tx \perp Ty$. Therefore T is $\perp$-preserving. For each $\perp$-comparable elements $x, y \in X$ with $x \neq y$, observed that

$$\frac{1}{2s}d(x, Tx) < d(x, y) \;\Leftrightarrow\; (x = 1 \wedge y \in Q) \vee (x \in Q \wedge y = 1).$$

Now, we consider three cases as follows.

(B_1) If $x = 1$ and $y \neq -6$, then $d(Tx, Ty) = 0$.

(B_2) If $x = 1$ and $y = -6$, then $d(Tx, Ty) = 1$ and $M_T(x, y) = d(Tx, y) + d(y, Ty) \cong 18.3$.

(B_3) If $x = -6$ and $y = 1$ then $d(Tx, Ty) = 1$ and $M_T(x, y) = d(x, Tx) \cong 9.85$.

Taking $F = \ln(t)$ and $\psi(t) = \dfrac{t}{20}$. Therefore, for all $\perp$-comparable $x, y \in X$ with $x \neq y$, we have $F(4s^5 d(Tx, Ty)) \leqslant F(M_T(x, y)) - \psi(M_T(x, y))$. Hence, applying Theorem 2.2 and Remark 2.3, T has a unique fixed point.

3. Some Consequences

In this section, we consider some special cases, where in our result deduce several well-known fixed point theorems of the existing literature.

Similarly, applying the steps in the proof of Theorem 2.2, we obtain following results.

Corollary 3.1. *Let $(X, \perp, d)$ be a SO-b-complete with constant s and X be $\perp$-regular. Also, let $T : X \to X$ be a self-mapping, SO-b-continuous and preserving and there exists $\tau > 0$ such that, for all $\perp$-comparable $x, y \in X$,*

$$\frac{1}{2s} d(x, Tx) < d(x, y) \quad \Rightarrow \quad \tau + F(4s^5 d(T^p x, T^p y)) \leqslant F(M_T(x, y)),$$

where where $M_T(x, y)$ is defined in Definition 1.5. Then T has a unique fixed point $x^* \in X$.

Corollary 3.2. *Let $(X, \perp, d)$ be a SO-b-complete with constant s and X be $\perp$-regular. Also, let $T : X \to X$ be a self-mapping, SO-b-continuous and preserving such that, for all $\perp$-comparable $x, y \in X$,*

$$\frac{1}{2s} d(x, Tx) < d(x, y) \quad \Rightarrow \quad F(4s^5 d(T^p x, T^p y)) \leqslant F(M_T(x, y)) - \psi(N(x, y)),$$

where

$$\begin{aligned} N_T(x, y) = \max\Big\{ & d(x, y), d(x, Tx), d(y, Ty) \\ & \frac{d(x, Ty) + yd(y, T)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, \\ & d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) \Big\}, \end{aligned}$$

and ψ is defined as in Theorem 1.6*. Then T has a unique fixed point $x^* \in X$.*

Corollary 3.3. *Let $(X, \perp, d)$ be a SO-b-complete with constant s and X be $\perp$-regular. Also, let $T : X \to X$ be a self-mapping SO-b-continuous and $\perp$-preserving such that, for every $x, y \in X$,*

$$\frac{1}{2s} d(x, Tx) < d(x, y) \quad \Rightarrow \quad F(4s^5 d(T^p x, T^p y)) \leqslant F(M_T(x, y)) - \psi(d(x, y)),$$

where $M_T(x, y)$ is defined in Definition 1.5 and ψ is defined as in Theorem 1.6*. Then T has a unique fixed point $x^* \in X$.*

4. Application to the Nonlinear Fractional Boundary Value Problems

In this section we give an application of our main results to a nonlinear fractional boundary value problem. For this let $X = C[0, 1]$ endowed with the metric d induced by

$$d(u, v) = \sup_{t \in [0,1]} |u(t) - v(t)|^2.$$

Thus, (X, d) is a b-metric space with $s = 2$. Consider the following fractional differential equations boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t)), & t \in [0, 1],\ 3 < \alpha \leqslant 4, \\ u(0) = u(1) = u^{'}(0) = u^{'}(1) = 0, \end{cases} \tag{25}$$

where D^{α}_{0+} is the standard Riemann-Liouville derivative and $f \in C[0, 1]$ such that

(C_1) For all $t \in [0, 1]$ and $u \in X$, $f(t, u)$ is increasing related to the second variable;

(C_2) For all $t, t^{'} \in [0, 1]$ and $u, v \in X$ with $u(t)v(t^{'}) \leqslant \max\{v(t), v(t^{'})\}$, we have

$$f(t, u(t))f(t^{'}, v(t^{'})) \leqslant \{f(t, u(t)v(t)) \ \ or\ \ f(t^{'}, u(t^{'})v(t^{'}))\};$$

(C_3) For all $u, v \in X$ with $u(t)v(t) \leqslant v(t)$ and $t \in [0, 1]$, we have

$$|f(t, u(t)) - f(t, v(t))| \leqslant \frac{1}{16}Q(u, v)^{1/2},$$

where $Q(u, v) := \max\{|u - v|^2, |T^2u - v|^2, \frac{|Tu - v|^2+|u - Tv|^2}{4}, \frac{|T^2u - u|^2+|T^2u - Tv|^2}{4}, |T^2u - Tv|^2+|T^2u - Tu|^2, |T^2u - Tv|^2+|Tu - u|^2, |Tu - v|^2+|v - Tv|^2\}$.

Theorem 4.1. *Let the above conditions are satisfied. Then problem* (25) *has a unique solution.*

Proof. We define the operator equation $T : X \to X$ as follows:

$$Tu(t) = \int_0^1 G(t, s)f(s, u(s))ds, \tag{26}$$

where

$$G(t, s) = \begin{cases} \frac{(1 - t)^{\alpha-1} + [(1 - s)t]^{\alpha-2}[(s - t) + (\alpha - 2)(1 - t)s]}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1 \\ \frac{(1 - s)^{\alpha-2}t^{\alpha-2}[(s - t) + (\alpha - 2)(1 - t)s]}{\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

The Greens function $G(t, s)$ defined above has the following property, for all $t, s \in [0, 1]$

$$G(t, s) \leqslant \frac{M_0K(s)}{\Gamma(\alpha)},$$

where $M_0 = \max\{\alpha - 1, (\alpha - 2)^2\}$ and $k(s) = s^2(1-s)^{\alpha-2}$. So, we conclude that

$$G(t,s) \leqslant 1, \quad \textit{for all } t, s \in [0,1]. \tag{27}$$

On the other word, we know that fractional differential equations boundary value problem has a unique solution if and only if T has a unique fixed point. We consider the following orthogonality relation in X:

$$u \perp v \quad \Leftrightarrow \quad u(t)v(t^{'}) \leqslant \max\{v(t), v(t^{'})\},$$

for all $t, t^{'} \in [0,1]$ and $u, v \in X$. Clearly, X is complete with the metric "d" defined above, then it is SO-b-complete. Also, from definition, X is $\perp$-regular. Now, we prove the following steps to complete the proof.

Step 1. T is $\perp$-preserving. Let $u, v \in X$ with $u \perp v$. We must show that $Tu(t)Tv(t^{'}) \leqslant max\{T(v(t)), T(v(t^{'}))\}$, for all $t, t^{'} \in [0,1]$ and $u, v \in X$. Applying (C_2), we have two cases:

(1). $f(s, u(s))f(s^{'}, v(s^{'})) \leqslant f(s, u(s)v(s))$ for all $s, s^{'} \in [0,1]$. Applying (26), (27) and (C_1), for all $t, t^{'} \in [0,1]$, we have

$$\begin{aligned}
Tu(t)Tv(t^{'}) &= \int_0^1 \left[\int_0^1 G(t,s)G(t^{'},s^{'})f(s,u(s))f(s^{'},v(s^{'}))ds\right] ds^{'} \\
&\leqslant \int_0^1 \left[\int_0^1 G(t,s)G(t^{'},s^{'})f(s,u(s)v(s))ds\right] ds^{'} \\
&\leqslant \int_0^1 \left[\int_0^1 G(t,s)f(s,u(s)v(s))ds\right] ds^{'} \\
&\leqslant \int_0^1 \left[\int_0^1 G(t,s)f(s,v(s))ds\right] ds^{'} \\
&= \int_0^1 G(t,s)f(s,v(s))ds \\
&= T(v(t)) \leqslant \max\{T(v(t)), T(v(t^{'}))\}.
\end{aligned}$$

(2). $f(s, u(s))f(s^{'}, v(s^{'})) \leqslant f(s^{'}, u(s^{'})v(s^{'}))$ for all $s, s^{'} \in [0,1]$. Applying (26), (27) and (C_1), for all $t, t^{'} \in [0,1]$, we have

$$\begin{aligned}
Tu(t)Tv(t^{'}) =& \int_0^1 \left[\int_0^1 G(t,s)G(t^{'},s^{'})f(s,u(s))f(s^{'},v(s^{'}))ds\right] ds^{'}\\
\leqslant& \int_0^1 \left[\int_0^1 G(t,s)G(t^{'},s^{'})f(s^{'},u(s^{'})v(s^{'}))ds\right] ds^{'}\\
\leqslant& \int_0^1 \left[\int_0^1 G(t^{'},s^{'})f(s^{'},u(s^{'})v(s^{'}))ds\right] ds^{'}\\
\leqslant& \int_0^1 \left[\int_0^1 G(t^{'},s^{'})f(s^{'},v(s^{'}))ds\right] ds^{'}\\
\leqslant& \int_0^1 G(t^{'},s^{'})f(s^{'},v(s^{'}))ds^{'}\\
=& T(v(t^{'})) \leqslant \max\{T(v(t)), T(v(t^{'}))\}.
\end{aligned}$$

These imply that T is $\perp$-preserving.

Step 2. We Show that there exists $\psi \in \Psi$ and $F \in \mathfrak{F}_G$ such that for each $\perp$-comparable elements $u, v \in X$ with $u \neq v$

$$d(u,Tu) < d(u,v) \ \Rightarrow\ F(4s^5 d(Tu,Tv)) \leqslant F(M_{_T}(u,v)) - \psi(M_T(u,v)).$$

For this purpose, applying (C_3), we have

$$\begin{aligned}
|Tu(t) - Tv(t)|^2 =& |\int_0^1 G(t,s)f(s,u(s))ds - \int_0^1 G(t,s)f(s,v(s))ds|^2\\
\leqslant& \left[\int_0^1 |G(t,s)||f(s,u(s)) - f(s,v(s))|ds\right]^2\\
\leqslant& \left[\frac{1}{16}Q(u,v)^{1/2}\int_0^1 |G(t,s)|ds\right]^2\\
\leqslant& \left[\frac{1}{16}Q(u,v)^{1/2}\right]^2\\
=& (\frac{1}{2})^8 Q(u,v) \leqslant (\frac{1}{2})^8 M_T(u,v).
\end{aligned}$$

We consider the definition of d, we have $d(Tu(t), Tv(t)) \leqslant (\frac{1}{2})^8 M_T(x,y)$, and so we have

$$\ln[2^7 d(Tu(t), Tv(t))] \leqslant \ln M_T(u,v) - \ln 2.$$

We set $F(t) = \ln(t)$ and $\tau = \ln 2$. Applying Corollary 3.1, T has a unique fixed point in X which is a unique solution of fractional differential equations boundary value problem (25). □

Remark 4.2. *By previous results, we can not guarantee the accuracy of Theorem 4.1 because conditions* (C_1) *and* (C_1) *does not apply to all* $u, v \in X$*, it applies only on limited numbers of them.*

5. Acknowledgements

The authors are indebted to the anonymous referees for the careful and competent reading of the present paper and for their valuable suggestions. They would like to express heartily thank to the associated editor, Dr. Mazyar Zarepour, for his supports during the review process.

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