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Features of the Efficiency Frontier and its Application in Inverse DEA Without Solving a Model

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Abstract. The inverse data envelopment analysis is an inverse optimization problem, which can be used as an appropriate planning tool for management decisions. The typical DEA mainly focuses on post-operative evaluation of an organizational performance. Sometimes economic conditions such as economic prohibitions on exports or imports are imposed on a system. These prohibitions prevent decision-making units from best performance (efficiency one). In this case, if the system has the best performance (with a less than one efficiency score) then it will be considered as an efficient system. So, the efficiency frontier changes problem must be studied. So by making change in definition of the best efficiency amount of a system, it still has the best performance. In these situations, the inefficient units can select a real pattern instead of reaching an unrealistic pattern that is presented in ideal terms to achieve the best conditions (the best efficiency value is one). So a long-term management plan can be developed. The efficiency frontier change will be expressed inputs and outputs as a coefficient of efficiency. The

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frontier change looks at the changes in inputs and outputs to reach the new frontier. One of the purposes of the data envelopment analysis is the investigation of inputs and outputs amounts by changing the amount of efficiency. So far, many models must be solved to calculating these changes. Efficiency frontier problem can replace a simple mathematical model with these models. All of these advantages can improve calculating input and outputs changes and RTS will be unchanged and decision maker can estimate units RTS without solving any model. So a unit will be stayed MPSS by reducing inputs. In other frontier change methods some hyperplanes and extreme units had been deleting but our method transforms them on new frontier. So all extreme units and RTS can estimate easily. The efficiency frontier changes can delete some inefficient units so systems cost will be reduced. For this purpose, in this paper, the change in the efficiency frontier, its properties and its effect on the inverse data envelopment analysis is examined.

AMS Subject Classification: 47N10
Keywords and Phrases: Data envelopment analysis(DEA), inverse data envelopment analysis, decision making units(DMU), return to scale (RTS), frontier changes, efficiency, input, output, production possibility set (PPS)

1. Introduction

Data Envelopment Analysis has been a standard tool for evaluating the relative efficiency. The efficiency of a unit is a function of various factors, such as the number of units (n), the input and output units (x, y), the number of input and output components (m, s), the type of technology of PPS. In some studies, researchers have estimated some of the affecting parameters on efficiency, such as the units inputs or outputs to maintaining or improving efficiency. This category of issues has been studied on performance analysis, is called Inverse Data Envelopment Analysis. The inverse DEA was introduced by Wei and et. al [13] in 2000. In their paper, they solved a question that how the output of a unit increases in a group of decision-making units if certain inputs of this unit is increased, so that the amount of efficiency remains unchanged. Wei et. al [13] introduced a multi-objective linear programming model for estimating the output level when the decision-making unit is low-yield, and when the decision making unit has a weak efficiency. Jahanshahloo et. al [6, 7, 8] generalized the question of Wei that if in a group of decision-making units some inputs of a under evaluation unit would be increased to a certain level, then how can the outputs of this unit increase, so that the unit's efficiency level be unchanged or improved. They answered this by introducing the necessary corrections in the previous models. Yan et. al [17] presented a model for estimating inputs and outputs. In 2008, Hadi Venchech et. al [5] corrected These conditions also they issued the problem of estimating inputs. They expressed in a group of decision-making units if certain outputs of under evaluation unit were increased to a certain level, then the inputs of this unit would increase. In other words, how much additional resources should be allocated to this unit, so that the unit's efficiency level remains at the same. In 2011, Lertworasrikul et. al [10] introduced a nonlinear programming problem in inverse data envelopment analysis model under variable-return to scale in the process of solving the model and then they used a Multi-objective programming structure to solve it. In 2015, ghiyasi [4] introduced a multi-objective programming problem. In their method a unique optimal solution is not necessary condition. This model made a modification to the model of Lertworasrikul [10]. In 2015, Jahanshahloo et. al [6] reviewed the inverse data envelopment analysis, in the temporary dependency of using multi-objective programming problems. Lim in 2016 [11] used frontier changes in the inverse optimization. Most inverse data envelopment analysis models are based on this fact that the efficiency amount of DMUs must be unchanged. By using economic concepts such as return-to-scale is used in the inverse-data envelopment analysis, it is clear that assumptions such as fixed-efficiency amount are false. Ebrahimkhani et. al [3] investigated frontier changes. But in this method some models were applied and some of the hyperplanes was deleted. RTS can estimate by using hyperplanes so in this method RTS cannot estimate correctly and it make complex calculations. In this paper we focus on the efficiency frontier problem to analyze its properties, including the position of extreme efficient units, creative hyperplanes and returns to scale that all of them are important points in managerial decisions. The efficiency frontier changes and its relationship with inputs and outputs changes is the basis of inverse DEA. A correct definition of it can eliminates the use of any model to compute changes in inverse DEA model. Accordingly, this paper contains 3 sections. First the inverse DEA is introduced. Then the proposed method is described with theorems and examples and at the end a numerical example will be express.

2. Inverse DEA [15]

Consider a set of decision-making units that $\theta_1, \cdots, \theta_n$ are their efficiency indicators. Suppose that inputs of DMU_0 increase to Δx. The goal is to calculate the maximum changes in outputs of DMU_0, (δy), so that the efficiency of DMU_0 stays constant. The capital prediction model (IPM) is used to find the maximum Δy such that the optimal value is $\theta_{n+1} = \theta_0$. In this case, Δy is an unspecified variable vector and θ_0 is the optimal value obtained by solving the CCR model by an additional DMU. The input and output vector of this DMU is $(X + \Delta x, Y + \Delta y)$. On the other hand, there is a weighted vector for the m inputs as $w = (w_1 \ \cdots \cdot \ \cdot w_n)^T$. Therefore, the IPM model is defined as:

$$\begin{aligned} &IPM \quad \min \theta \equiv \theta_{n+1} \\ &\sum_{j=1}^{n} \lambda_j w^T X_j + \lambda_{n+1} w^T (X_0 + \Delta X) \leqslant \theta w^T (X_0 + \Delta X) \\ &\sum_{j=1}^{n} \lambda_j Y_j + \lambda_{n+1} (Y_0 + \Delta y) \leqslant (Y_0 + \Delta y) \\ &\lambda_j \geqslant 0, j = 1, 2, \ldots n, \end{aligned} \tag{1}$$

for solving the above model (1) an algorithm is presented, which lead to a multi-objective problem:

$$\begin{aligned} (MOP) \quad & \max(\Delta y_{10}, \ldots, \Delta y_{s0}) \\ \text{s.t} \quad & X\lambda \leqslant \theta_0 (X_0 + \Delta x) \\ & Y\lambda \geqslant Y_0 + \Delta y \\ & \lambda \geqslant 0 \end{aligned} \tag{2}$$

3. Suggested Method

In this section the efficiency frontiers change and its effects on RTS, hyperplanes and extreme efficient units will discuss.

3.1 Meaning of efficiency frontier change

Sometimes economic conditions, such as export or import bans, will change the inputs and outputs of a system. So any unit cannot have the best performance. In this situation, if the system has the best performance (the best performance is less than one), it is still an efficient system with the desired performance level. So efficiency frontier changes problem will be expressed. In this situation the effects of efficiency frontier change on extreme efficient units and returns to scale can study which is one of the main concepts in management decisions. First, let's illustrate efficiency frontier change with a simple example. If A and B be arbitrary units on the efficiency frontier, first their changes that are due to the efficiency frontier changes will be analyzed. Figure 1 shows the position of these units on the production possibility set.

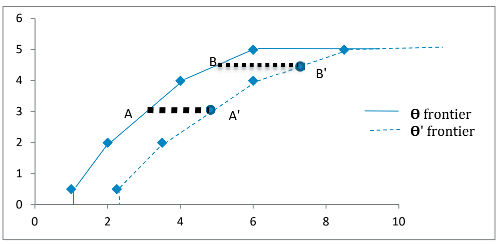

Figure 1. The changes of frontier and DMUs

As can be seen $x_{A'} = x_A + \alpha$, $y_{A'} = y_A$ and $x_{B'} = x_B + \beta$, $y_{B'} = y_B$, so:

$$\theta' = \tfrac{x_A}{x_A+\alpha} = \tfrac{x_B}{x_B+\beta} = \theta' \;\Rightarrow\; \begin{cases} \alpha = x_A \times \left(\frac{1-\theta'}{\theta'}\right) \\ \beta = x_B \times \left(\frac{1-\theta'}{\theta'}\right) \end{cases} \Rightarrow x_{A'} = \frac{1}{\theta'} x_A, x_{B'} = \frac{1}{\theta'} x_B$$

In other words, if frontier changes from θ to θ' then the coordinates of A $(DMU_A = (X_A, Y_A))$ will be change to $DMU_{A'} = \left(\frac{1}{\theta'} X_A, Y_A\right)$. In other words, all input components will be change and the output components will be constant.

3.2 Studying on extreme efficiency

The efficiency frontier change creates new extreme efficient units. At this point, the Anderson Peterson model [1] will be used to find these extreme efficient units. In other words, by changing the efficiency frontier, the previous extreme efficient units are transmitted to the new extreme efficient units on the new frontier.

Theorem 3.2.1. *If DMU_A be efficient on the θ frontier, we show that $DMU_{A'} = \left(\frac{1}{\theta'}X_A, Y_A\right)$ is an extreme efficient unit on the θ' frontier.*

Proof. Consider Anderson Peterson's Model. If new conditions is applied to all units, then for $DMU_{j'} = \left(\frac{1}{\theta'}X_j, Y_j\right)$, $j = 1, \ldots, n$, the model is written as follows:

$$\begin{aligned} &\min \varphi \\ \text{s. t} \quad &\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j \frac{1}{\theta'} x_{ij} \leqslant \varphi \frac{1}{\theta'} x_{i0} \\ &\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj} \leqslant y_{r0} \\ &\lambda_j \geqslant 0, \ j = 1, \ldots, n, \end{aligned}$$

so

$$\begin{aligned} &\min \varphi \\ \text{s. t} \quad &\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_{ij} \leqslant \varphi x_{i0} \\ &\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj} \leqslant y_{r0} \\ &\lambda_j \geqslant 0, \ j = 1, \ldots, n. \end{aligned}$$

Since the efficiency frontier is applied to the input vectors of all DMUs, so there is no change in the model constraints, and therefore extreme efficient units on the θ frontier will be extreme efficient unit on θ' frontier. $\square$

3.3 Creative hyperplanes

So far, it has become clear that the efficiency frontier changes will not make change in extreme efficient units. But it should be noted that in this situation creative hyperplane equations are passing of these extreme units how can be. First, discus these changes with an example. Consider figure 2:

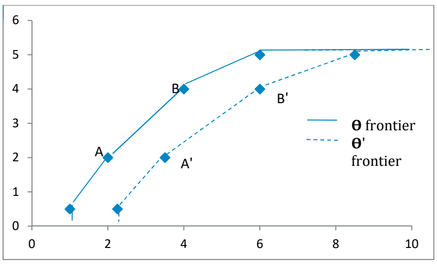

Figure 2. Investigating the efficiency frontier changes effect on the extreme efficient units

if $DMU_A = (X_A, Y_A)$, $DMU_B = (X_B, Y_B)$, $DMU_{A'} = (\frac{1}{\theta'}X_A, Y_A)$, $DMU_{B'} = (\frac{1}{\theta'}X_B, Y_B)$ then the line equation will be as follow:

$$Y - Y_{A'} = \frac{Y_{B'} - Y_{A'}}{X_{B'} - X_{A'}}(X - X_{A'}) \Rightarrow Y - Y_A = \frac{Y_B - Y_A}{\frac{1}{\theta'}(X_B - X_A)}\left(X - \frac{1}{\theta'}X_A\right)$$

$$\Rightarrow Y - Y_A = \theta'\left(\frac{Y_B - Y_A}{X_B - X_A}\right)\left(X - \frac{1}{\theta'}X_A\right) \Rightarrow Y - Y_A = \left(\frac{Y_B - Y_A}{X_B - X_A}\right)(\theta' X - X_A)$$

If $m = \frac{Y_B - Y_A}{X_B - X_A}$ so:

A) The line's equation between A and B on the θ frontier is: $Y - Y_A = m(X - X_A) \Rightarrow Y = mX + (-mX_A + Y_A)$

B) The line equation between A' and B' on the θ' frontier is: $Y - Y_A = m(\theta' X - X_A) \Rightarrow Y = \theta' mX + (-mX_A + Y_A)$

In other words, the new extreme efficient units just change the slope line and the width from the origin will be constant. In general, if A and B

be two extreme efficient points on the θ frontier, then the normal vector is perpendicular to the line, so:

$$N_1(X_A - X_B) + N_2(Y_A - Y_B) = 0 \;\Rightarrow\; N_1 = \frac{N_2(Y_A - Y_B)}{(X_A - X_B)}$$

Hyperplane equation will be as follow:

$$N_1(X - X_B) + N_2(Y - Y_B) = 0 \;\Rightarrow\; N_1X + N_2Y - (N_1X_B - N_2Y_B) = 0$$

$$\xrightarrow{\text{if } u_0 = N_1X_B - N_2Y_B} N_1X + N_2Y - u_0 = 0$$

Now consider these units to be transmitted to the θ' frontier, so that the new extreme efficient units are $DMU_{A'} = (\frac{1}{\theta'}X_A, Y_A)$, $DMU_{B'} = (\frac{1}{\theta'}X_B, Y_B)$. The changes of normal vector are as follows:

$$N_1\left(X_{A'} - X_{B'}\right) + N_2\left(Y_{A'} - Y_{B'}\right) = N_1\left(\frac{1}{\theta'}X_A - \frac{1}{\theta'}X_B\right) + N_2\left(Y_A - Y_B\right)$$

$$= \tfrac{1}{\theta'}N_1(X_A - X_B) + N_2(Y_A - Y_B) = 0 \Rightarrow\; N_1 = \frac{N_2\theta'(Y_A - Y_B)}{(X_A - X_B)}$$

$$\Rightarrow\; N_1' = \theta' N_1,\; N_2' = N_2$$

Creative hyperplane equation is:

$$N_1'(X - X_{B'}) + N_2'(Y - Y_{B'}) = \theta' N_1\left(X - \frac{1}{\theta'}X_B\right) + N_2(Y - Y_B) = 0$$

$$\theta' N_1X + N_2Y - (N_1X_B - N_2Y_B) = 0 \;\xrightarrow{\text{if } u_0 = N_1X_B - N_2Y_B}\; \theta' N_1X + N_2Y - u_0 = 0$$

As can be seen $\theta' < 1$, then $N_1' = \theta' N_1 < N_1$ and therefore:

$$N_1'X < N_1X \;\Rightarrow\; N_1'X + N_2Y - u_0 < N_1X + N_2Y - u_0$$

Which indicates by changing the efficiency frontier, the PPS moves to the right and becomes more restrictive.

3.4 The returns to scale

As stated in the previous section, the width of the origin from the creative hyperplane will be change by changing the efficiency frontier. One of the important issues in economic is the problem of returns to scale. Now, the efficiency frontier changes effects on RTS will be examined.

Theorem 3.4.1. *By changing the efficiency frontier, the returns to the scale of the decision-making units will be changed.*

Proof. The hyperplane equation is $N_1X + N_2Y - u_0 = 0$ that is as $\theta' N_1X + N_2Y - u_0 = 0$ on the new frontier. On the other hand, one of the methods to estimating the return to scale is the u_0 method. As the hyperplane equation shows, the change of frontier will change the width of the origin of the creative hyperplane to $\frac{u_0}{\theta'}$ and the output component of the normal vector will be changed to $\frac{N_2}{\theta'}$. So the type of returns to the scale will be changed. □

3.5 Using the concept of efficiency frontier change in the inverse data envelopment

All methods that have been presented to evaluating the amount of input or output's changes by changing in efficiency are based on the use of two or more models. These multi-model methods will be time-consuming also make computational complexity. By using simple mathematical calculations, the frontier change concept provides the conditions for calculating the amount of change in input or output.

3.5.1. Estimating the changes of inputs for a change in the efficiency value from θ to θ'

If the change in the input vector of DMU_0 is represented by Δx_0, then by changing the efficiency value of DMU_0 from θ to θ', the new input vector is calculated as follows:

$$\gamma = \frac{\theta'}{\theta} \;\Rightarrow\; X_{0\text{new}} = X_0 + \Delta x_0 = \frac{1}{\gamma}X_0$$

Note that the output vector is considered unchanged. In other words, the efficiency of $\left(\frac{1}{\gamma}X_0, Y_0\right)$ is θ'.

3.5.2. Estimating the output changes for a change in the efficiency value from θ to θ'

If the change of the output vector of DMU_0 is represented by Δy_0, then by changing the efficiency of DMU_0 from θ to θ', the new output vector is calculated as follows:

$$\gamma = \frac{\theta'}{\theta} \;\Rightarrow\; Y_{0\text{new}} = Y_0 + \Delta y_0 = \gamma Y_0$$

Note that the input vector is considered unchanged. In other words, efficiency of $(X_0, \gamma Y_0)$ will be θ'.

Theorem 3.5.2.1. *If DMU_0 be an extreme efficient unit, then by changing the efficiency from θ to θ' we have, $X_{0\text{new}} = X_0 + \Delta x_0 > \frac{1}{\gamma} X_0$ and $Y_{0\text{new}} = Y_0 + \Delta y_0 < \gamma Y_0$.*

Proof. By changing the input or output components or reducing the efficiency in an extreme efficient unit, not only the coordinates of this unit but also the PPS will change. This PPS limitation will shift the frontier. If a decision maker wants to reach the desired efficiency so the amount of change in input should be greater than the obtaining changes, and the amount of change in output is less than the obtaining output. □

3.5.3. Estimating the efficiency changes

So far, the changes been non radial in many of the inverse DEA models. Assume that the input vector of DMU_0 , is multiplied in γ, then θ_{new} is calculated as follows:

$$X_0 \longrightarrow \gamma X_0 \;\Rightarrow\; \theta_{0\text{new}} = \frac{1}{\gamma}\theta_0$$

In other words, changes in inputs will change the amount of efficiency in reverse.

Theorem 3.5.3.1. *If DMU_0 be an extreme efficient unit, then it will be changed in the input components $\theta_{0\text{new}} > \dfrac{1}{\gamma}\theta_0$*

Proof. By changing the input or output components of an extreme efficient unit, not only the coordinates of this unit but also the PPS will be changed. This PPS limitation will shift the frontier. Therefore, the amount of efficiency change should be greater than the obtaining changes. □

4. Numerical Example

4.1 An illustrative example

Consider the five decision making units as presented in figure (3) and table (1):

Table 1: Input, output and efficiency of these DMUs

	A	B	C	D	E
Input	1	2	4	6	5
Output	0.5	2	4	5	3
Efficiency	1	1	1	1	0.6

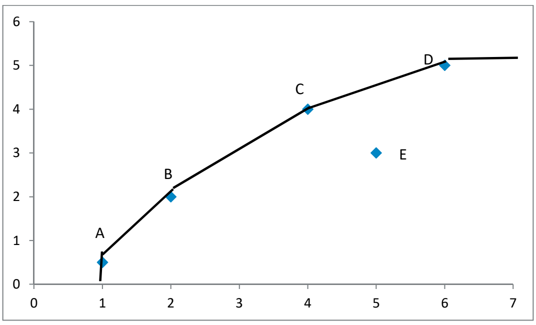

Figure 3. Position of DMUs

Consider the inefficient unit as E. The conditions of E are studied in two situations.

A) Increasing the efficiency and calculating the amount of changes in input and output

If the efficiency amount of this unit changes from 0.6 to 0.8 then the new input and output will be as follows:

$$\gamma = \frac{0.8}{0.6} = \frac{4}{3} \Rightarrow \begin{cases} X_{\text{new}} = \frac{1}{\gamma}X = 5 \times 0.75 = 3.75 \\ Y_{\text{new}} = \gamma Y = 3 \times \frac{4}{3} = 4 \end{cases}$$

The efficiency value of all units after the changing of the input and output vector of E is shown in Table (2).

Table 2: The efficiency changes of DMUs

	A	B	C	D	E
Efficiency after Changes in inputs of E	1	1	1	1	0.8
Efficiency after Changes in output of E	1	1	1	1	0.8
Efficiency with out changes	1	1	1	1	1

The position of E after an input change is in figure (4) and the status of unit E and the new frontier after an output change is shown in figure (5).

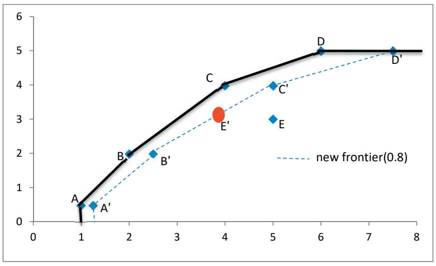

Figure 4. New frontier after a change in efficiency of DMU_E

As can be seen, with the change in the input vector of unit E, this unit is exactly on the new frontier so its efficiency amount has been changed to 0.8 and E is an efficient unit on the new frontier.

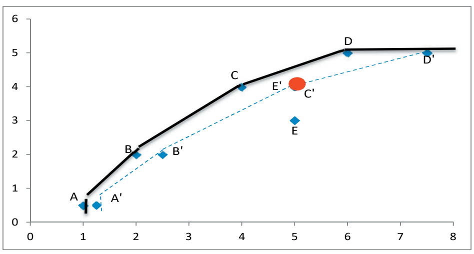

Figure 5. New frontier after a change in efficiency of DMU_E

As shown in Fig (5), the change in the output vector of Unit E transforms this unit into an extreme efficient unit on the new frontier. In other words, it can be said that by an increasing at the output of unit E, this unit will be converted into an extreme efficient unit on new frontier.

According to the above example, a change in the input or output vectors of an unit is converted this unit into an efficient unit or an extreme efficient unit on the new frontier.

B) A change in input and this effect on calculating the efficiency's change:
Suppose that the inputs of the unit E are tripled. The new efficiency based on the proposed method will be as follows:

$$\gamma = 3,\ X_{\text{new}} = 5 \times 3 = 15 \ \Rightarrow\ \theta_{\text{new}} = \frac{1}{3} \times 0.6 = 0.2$$

The new frontier and position of unit E on it is shown in figure (6). Table (3) shows the efficiency of all DMUs by solving the model.

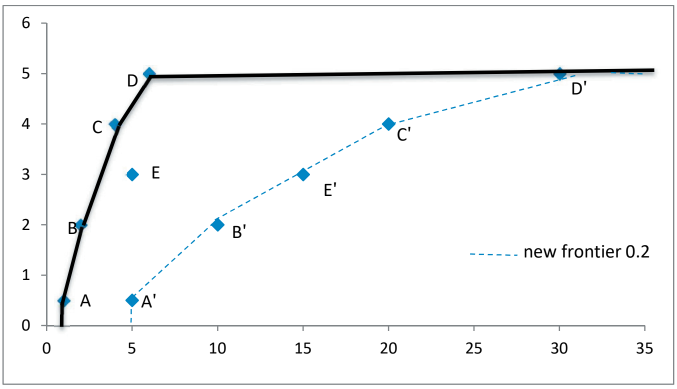

Figure 6. New frontier after a change in efficiency of DMU_E

As shown in Figure (6), after the change in input of unit E, this unit will be on 0.2 frontier. That means its efficiency score is 0.2. It must be noted that E' is transmitted of E.

4.2 Example

Information of twenty bank branches in IRAN and their efficiency and extreme efficiency are presented in Table (3). In this example, there are three inputs (concessionaire, received deposits and the benefit of the guarantee) and two outputs (profit received and the commission received).

Table 3: Inputs and outputs of bank barnchs

	Input 1	Input 2	Input 3	Output 1	Output 2	Efficiency	Cross efficiency
1	653640	4369768	1172377	385951	21031	1	1.37
2	3414100	2412669	53877	581119	654	0.76	0.76
3	17711	835433	31566	6258	276	0.79	0.79
4	5959869	2600965	103682	1949549	1500	1	1.61
5	141245	1631590	17085	37385	153	0.64	0.64
6	52862	2730464	59489	27989	827	0.79	0.79
7	8995	654427	3561	2143	16	1	2.69
8	178055	1478011	59565	119756	831	1	1.12
9	1093092	1805003	1376960	914956	24656	1	2.77
10	39936	966818	4792	45183	36	1	2.79
11	14188	959469	253275	8148	4592	1	6.83
12	192641	989709	28197	56509	41	0.7	0.7
13	12476	698185	137267	3541	2099	1	1.12
14	80076	683408	23008	40555	335	1	1.02
15	1368245	1798316	140641	350100	1036	0.66	0.66
16	67191	548403	16208	20472	194	1	1.21
17	7625	2629956	14142	2246	231	1	1.21
18	91748	1374525	188351	15782	1940	0.57	0.57
19	780324	1375046	442314	201537	7606	0.95	0.95
20	44933090	2931475	114434	8564575	1344	1	3.98

Suppose that the efficiency value of an inefficient unit like DMU_{18} is increased from 0.57 to one. In this case $\gamma = \frac{1}{0.57} = 1.75 \rightarrow \frac{1}{\gamma} = 0.57$. So we have:

$$X_{18\text{new}} = (91748 \times 0.57, 1374525 \times 0.57, 188351 \times 0.57) = (52296.36, 783479.25, 107360.07)$$

With the new input and output vector $(X_{18}, Y_{18\text{new}})$, the efficiency of this unit will be one. Also, the new output to achieve the one efficiency is as follows:

$$Y_{18\text{new}} = (15782 \times 1.75, 1940 \times 1.75) = (27618.5, 3395)$$

When the input and output vector is $(X_{18}, Y_{18\text{new}})$, the unit 18 will be achieved the one score efficiency without any change the inputs. In other words, with proper variations in the input or output vectors, an inefficient unit can be converted into an efficient unit without solving any model.

Now suppose the inputs of unit 18, is multiplied by 0.625. Then, the new input vector is $X_{18\text{new}} = (57312.58, 59078.1251, 17719.375)$. It is expected that the new unit's efficiency be equal to $\frac{1}{0.625} = 1.6$. That means, if the inputs multiplied by 0.625 then the efficiency will be 0.91. Now consider an extreme efficient unit like DMU_4. If the efficiency of this unit decreases from 1 to 0.8 so based on the theorem (3.5), the changes in the inputs and outputs will be as follows.

$$X_{4\text{new}} > (5959869 \times 1.25, 2600965 \times 1.25, 103682 \times 1.25) = (7449836.25, 3251206.25, 129602.5)$$

$$Y_{4\text{new}} < (1949549 \times 0.8, 1500 \times 0.8) = (1559639.2, 1200)$$

The obtained values from solving the model (1) for inputs and outputs are as follows:

$$X_{4\text{new}} = (744936, 4090932, 1593649) > (7449836.25, 3251206.25, 129602.5)$$

$$Y_{4\text{new}} = (1510788.4, -437260.6) < (1559639.2, 1200)$$

The results are quite consistent with presented model. This changes in values is due to the change in PPS that is created by extreme efficient units. In fact, a change in an extreme efficient unit causes a change in the PPS frontier so the resulting values will not be real. Similarly, the status of all decision-making units after a change can be examined and used to analyze a system and other decision-making units.

Conclusion

Inverse DEA studies is based on the unchanging efficiency of the observed decision-making units (DMUs) which shows the limitation of the inverse DEA for a sensitivity analysis. On the other hand, it allows the input and / or output substitution levels to produce the same efficiency score. The efficiency frontier change is one of the topics that discussed in data envelopment analysis and can be used as a powerful tool for analyzing inverse DEA. By using the frontier changes, we dont need any model to analyze the sensitivity of decision-making units. Also, it provides the unique features such as maintaining the position of extreme efficient units and making a long term management plans. The frontier change makes a linear relationship between outputs and efficiency and a reverse relationship between efficiency and inputs. By these relationships we dont need to other models and replace them with simple mathematical calculations. The obtained values will be converted under evaluation unit into an efficient or an extreme efficient unit. In other words, if by changing the inputs or outputs, an inefficient unit is converted to an extreme efficient unit since the change in the frontier maintains the extreme efficiency property, so this unit will be extreme efficient on any frontier. In other words, the frontier change problem shows the least amount of variation in a unit to be on the ideal performance path.

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