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A Note on Maximal Numerical Range

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Abstract. We shed some light on the Pythagorean relation for operator established in [7] and we study its relationship with the maximal numerical range. We then get some informations of maximal numerical ranges of selfadjoint operators. This allows us to show, contrary to the closure of the numerical range, the non-continuity in the sens of Hausdorff of the maximal numerical range.

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1. Introduction

First, let us set some notation and terminology.
Let L be a subset of the complex plane $\mathbb{C}$. As usual, the symbols $\overline{L}$ and $co(L)$ stand for the closure and the convex hull of L, respectively. By an operator we throughout the note understand a bounded linear operator acting on an infinite dimensional complex Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, the numerical range of A is defined by the formula

$$W(A) = \langle Ax, x\rangle : x \in \mathcal{H}, \ \|x\| = 1,$$

where $\langle ., .\rangle$ and $\|.\|$ stand, respectively, for the inner product on $\mathcal{H}$ and the norm associated with it. It is a celebrated result due to Toeplitz and Hausdorff that $W(A)$ is a bounded convex set in the complex plane, see [5]. It is closed if $dim(\mathcal{H}) < \infty$, but it is not always closed if $dim(\mathcal{H}) = \infty$.

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For $A \in \mathcal{B}(\mathcal{H})$, let $\sigma(A)$, $r(A)$ and $w(A)$ denote the spectrum, the spectral radius and the numerical radius of A respectively and defined as follows

$$\sigma(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \ \text{ is not invertible}\},$$

$$r(A) = \sup\{|z| : \ z \in \sigma(A)\} \ \text{ and } \ w(A) = \{\sup |z| : \ z \in W(A)\}.$$

It is well known that $\sigma(A)$ is a compact set and $co(\sigma(A)) \subseteq \overline{W(A)}$.

For more material about the specral radius, the numerical radius and other information on the basic theory of algebraic numerical range, we mention here the books [2, 3, 4, 5] as standard sources of references.

It is a basic fact that $w(.)$ is a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the C^*-norm $\|.\|$. In fact, for any operator $A \in \mathcal{B}(\mathcal{H})$, the following inequalities are well known

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|.$$

An operator $A \in \mathcal{B}(\mathcal{H})$ is called normaloid if $w(A) = \|A\|$ or equivalently $r(A) = \|A\|$, see [4, Theorem 1.3-2]. Familiar examples of normaloid operators are normal operators; those A for which $A^*A = AA^*$, where A^* is the adjoint of A. There is another set that is close to $W(A)$; that is the maximal numerical range $W_0(A)$ of A. It it was introduced by Stampfli [7] and defined by

$$W_0(A) = \{\lim_n \langle Ax_n, x_n\rangle : x_n \in \mathcal{H}, \ \|x_n\| = 1, \ \lim_n \|Ax_n\| = \|A\|\}.$$

It is shown in [7, Lemma 2] that $W_0(A)$ is nonempty, closed, convex, and contained in the closure of the numerical range; $W_0(A) \subseteq \overline{W(A)})$. Then, $w_0(A) \leqslant w(A)$, where

$$w_0(A) = \sup\{|z| : \ z \in W_0(A).$$

It is also shown in [7, Corollary of Theorem 2] that there exists a unique scalar c_A (called center of mass of A) satisfying the following (*Pythagorean relation for operator*)

$$\|A - c_A\|^2 + |\lambda|^2 \leqslant \|(A - c_A) + \lambda\|^2 \text{ for all } \lambda \in \mathbb{C} \tag{1}$$

and $0 \in W_0(A - c_A)$. In particular, taking $\lambda = c_A$ in the inequality 1, we get

$$\|A - c_A\|^2 + |c_A|^2 \leqslant \|A\|^2. \tag{2}$$

Note also that from the inequality 1, we obtain

$$\|A - c_A\| = \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|.$$

In Section 2, we show that for any operator $A \in \mathcal{B}(\mathcal{H})$ we have $|c_A| \leqslant w_0'(A)$ where

$$w_0'(A) = \inf\{|z| : \ z \in W_0(A)\}$$

and we give a geometric interpretation of this result in the case where $c_A \neq 0$. From this result, we deduce that

$$c_A \in W_0(A) \text{ if and only if } \|A - c_A\|^2 + |c_A|^2 = \|A\|^2. \tag{3}$$

Using the equivalence 3, we give some examples of maximal numerical ranges and we establish the non-continuity in the sens of Hausdorff of the maximal numerical range.

From now on, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators acting on an infinite dimensional complex Hilbert space $\mathcal{H}$.

2. Phytagorean Relation and Maximal Numerical Range

In this section, we shed some light on the maximal numerical range $W_0(A)$ of an operator $A \in \mathcal{B}(\mathcal{H})$. More precisely, we show that $w_0'(A) \geqslant |c_A|$. This gives, on the one hand, a refinement of the inequality $w(A) \geqslant |c_A|$ ($c_A \in \overline{W(A)}$, see [7]) and on the other hand some information of the part in $\overline{D}(O, \|A\|)$ (the closed disk of radius $\|A\|$ centered at the origin) where the maximal numerical range $W_0(A)$ is contained. We then deduced sufficient and necessary conditions to have $c_A \in W_0(A)$.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then, $w_0'(A) \geqslant |c_A|$.*

Proof. By an argument of compactness, there exists $\alpha \in W_0(A)$ such that $|\alpha| = w_0'(A)$. Hence, there is a sequence of unit vectors $x_n \in \mathcal{H}$ satisfying

$$\alpha = \lim_n \langle Ax_n, x_n \rangle \quad \text{and} \quad \lim_n \|Ax_n\| = \|A\|.$$

Therefore, we have

$$\begin{aligned}\|A-c_A\|^2 &\geqslant \|(A-c_A)x_n\|^2\\ &= \|Ax_n\|^2+|c_A|^2-2Re(\overline{c_A}\langle Ax_n,x_n\rangle)\\ &\geqslant \|Ax_n\|^2+|c_A|^2-2|c_A|\ |\langle Ax_n,x_n\rangle)|.\end{aligned}$$

It results that

$$\begin{aligned}\|A-c_A\|^2 &\geqslant \|A\|^2+|c_A|^2-2|c_A|w_0^{'}(A)\\ &= \|A\|^2-(w_0^{'}(A))^2+(w_0^{'}(A)-|c_A|)^2.\end{aligned}$$

Thus,

$$\|A-c_A\|^2+(w_0^{'}(A))^2 \geqslant \|A\|^2+(w_0^{'}(A)-|c_A|)^2.$$

We see that

$$\|A-c_A\|^2+(w_0^{'}(A))^2 \geqslant \|A\|^2 \tag{4}$$

and from the inequality 2, we get $w_0^{'}(A) \geqslant |c_A|$. $\square$

Geometric interpretation 2.2. Given an operator $A \in \mathcal{B}(\mathcal{H})$, from Theorem 2.1, $w_0^{'}(A) \geqslant |c_A|$. In geometric terms, if $c_A \neq 0$, then $W_0(A)$ is outside of the open disk $D(O,|c_A|)$. Being convex, $W_0(A)$ is contained in the intersection of a half plane and $\overline{D}(O,\|A\|)$ (gray area; see Figure 1).

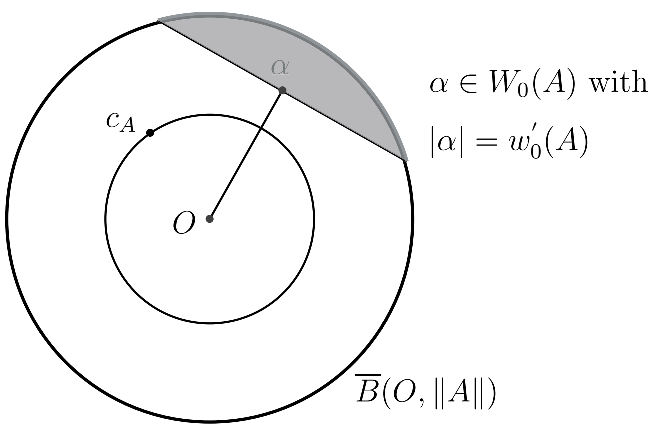

Figure 1. Geometric place of the numerical range

As cited above, we have $c_A \in \overline{W(A)}$. However, c_A need not be contained in $W_0(A)$. Indeed, the following corollary gives sufficient and necessary conditions to have $c_A \in W_0(A)$ (see Figure 2).

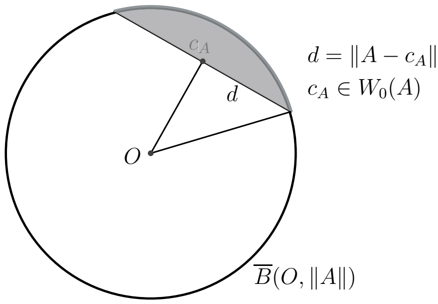

Figure 2. Phytagorean relation

Corollary 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then, the following are equivalent statements*

i) $c_A \in W_0(A)$,

ii) $w_0^{'}(A) = |c_A|$,

iii) $\|A - c_A\|^2 + |c_A|^2 = \|A\|^2$.

Proof. *i)* $\Rightarrow$ *ii)*. It results from Theortem 2.1.

ii) $\Rightarrow$ *iii)*. If $w_0^{'}(A) = |c_A|$, by the inequality (4), $\|A - c_A\|^2 + |c_A|^2 \geqslant \|A\|^2$. We deduce by the inequality (2) that $\|A - c_A\|^2 + |c_A|^2 = \|A\|^2$.

iii) $\Rightarrow$ *i)*. Assume $\|A - c_A\|^2 + |c_A|^2 = \|A\|^2$. Accoroding to Barraa-Boumazguour [1, Theorem 3.2]

$$\|A - c_A\|^2 = \sup_{x \in \mathcal{H},\ \|x\|=1} (\|Ax\|^2 - |\langle Ax, x\rangle|^2).$$

Then, there exists a sequence of unit vectors $x_n \in \mathcal{H}$ such that

$$\|A - c_A\|^2 = \lim_n (\|Ax_n\|^2 - |\langle Ax_n, x_n\rangle|^2). \tag{5}$$

We have

$$\begin{aligned}
\|A - c_A\|^2 &\geqslant \|(A - c_A)x_n\|^2 \\
&= \|Ax_n\|^2 + |c_A|^2 - 2Re(\overline{c_A}\langle Ax_n, x_n\rangle) \\
&\geqslant \|Ax_n\|^2 - |\langle Ax_n, x_n\rangle|^2 + |c_A - \langle Ax_n, x_n\rangle)|^2,
\end{aligned}$$

then from the equality (5), we get

$$\|A - c_A\|^2 \geqslant \|A - c_A\|^2 + |c_A - \mu|^2,$$

where $\mu = \lim_n \langle Ax_n, x_n\rangle$. It results that $c_A = \lim_n \langle Ax_n, x_n\rangle$ and the equality (5) becomes

$$\|A - c_A\|^2 + |c_A|^2 = \lim_n \|Ax_n\|^2,$$

and using the hypothesis, we conclude that $\lim_n \|Ax_n\| = \|A\|$. Consequently, $c_A \in W_0(A)$. $\square$

Remark 2.4. *From the previous proof, there is a sequence of unit vectors $x_n \in \mathcal{H}$ such that*

$$c_A = \lim_n \langle Ax_n, x_n\rangle \quad \text{and} \quad \|A - c_A\|^2 = \lim_n (\|Ax_n\|^2 - |\langle Ax_n, x_n\rangle|^2).$$

This is another way to see that $c_A \in \overline{W(A)}$ and also it provides another proof of uniqueness of the scalar c_A. Indeed, we can assume, without loss of generality, that $c_A = 0$. Suppose, for the sake of contradiction, that there is some scalar $\lambda_0 \neq 0$ satisfies

$$\|A - \lambda_0\| = \inf_{\lambda \in \mathbb{C}} \|A - \lambda\| = \|A\|. \tag{6}$$

Then, there would be a sequence of unit vectors $y_n \in \mathcal{H}$ with

$$\lambda_0 = \lim_n \langle Ay_n, y_n\rangle \quad \text{and} \quad \|A - \lambda_0\|^2 = \lim_n (\|Ay_n\|^2 - |\langle Ay_n, y_n\rangle|^2).$$

We would then have

$$\|A - \lambda_0\|^2 = \lim_n (\|Ay_n\|^2 - |\lambda_0|^2).$$

Since $\lambda_0 \neq 0$, it would follow that

$$\begin{aligned}\|A - \lambda_0\|^2 &< \lim_n \|Ay_n\|^2 \\ &\leqslant \|A\|^2.\end{aligned}$$

We obtain the derised contradiction from the equality (6).

As an application of the obtained results, we determine the maximal numerical ranges of some operators. We need the following lemma.

Lemma 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be normaloid. Then, $\sigma(A) \cap W_0(A)$ is a nonempty set. Moreover, for any $\lambda \in \sigma(A)$ with $|\lambda| = \|A\|$, we have $\lambda \in \sigma(A) \cap W_0(A)$.*

Proof. If A is a normaloid operator, then there exists $\lambda \in \sigma(A)$ such that $|\lambda| = \|A\|$. Since $\sigma(A) \subseteq \overline{W(A)}$, then there is a sequence of unit vectors $x_n \in \mathcal{H}$ such that $\lambda = \lim_n \langle Ax_n, x_n\rangle$. Using the Cauchy-Schwarz inequality, we have

$$\|A\| \geqslant \lim_n \|Ax_n\| \geqslant \lim_n |\langle Ax_n, x_n\rangle| = |\lambda| = \|A\|.$$

We derive that $\lim_n \|Ax_n\| = \|A\|$ and hence $\lambda \in W_0(A)$. The rest of the lemma is then obvious. $\square$

Example 2.6. Let A be a normal operator acting on the complex Hilbert space $\mathcal{H} = \mathbb{C}^2$ with $\sigma(A) = \{\lambda_1, \lambda_2\}$ and $|\lambda_1| \leqslant |\lambda_2|$. Then, since A is normal and $dim(\mathcal{H}) < \infty$, $W(A) = co(\sigma(A)) = [\lambda_1, \lambda_2]$; the closed line segment connecting λ_1 with λ_2. On the other hand, c_A is the center of the smallest disk containing $\sigma(A)$, see [7, Corollary 1 of Theorem 4], hence $c_A = \dfrac{\lambda_1 + \lambda_2}{2}$. We can easily check that $\|A - c_A\| = |\dfrac{\lambda_1 - \lambda_2}{2}|$ and therefore $\|A - c_A\|^2 + |c_A|^2 = \|A\|^2$ if and only if $|\lambda_1| = |\lambda_2|$. We have to discuss two cases.

First case: $|\lambda_1| = |\lambda_2|$. We have $\lambda_1, \lambda_2 \in \sigma(A)$ and $|\lambda_1| = |\lambda_2| = \|A\|$. Since A is normaloid, it results by Lemma 2.5 that $\lambda_1, \lambda_2 \in W_0(A)$. Being convex, $W_0(A) = [\lambda_1, \lambda_2]$.

Second case: $|\lambda_1| < |\lambda_2|$. We have $\|A - c_A\|^2 + |c_A|^2 \neq \|A\|^2$ and by Corollary 2.3, $c_A \notin W_0(A)$. But $\lambda_2 \in W_0(A)$, it follows that $W_0(A) \subset (c_A, \lambda_2]$. We claim that $W_0(A) = \lambda_2$. For this, suppose there is a scalar μ such taht $\mu \in W_0(A) \cap (c_A, \lambda_2)$. We can find a nonzero scalar α satisfying

$$\lambda_2 - \alpha \in (\mu, \lambda_2) \quad \text{and} \quad |\lambda_1 - \alpha| < |\lambda_2 - \alpha|.$$

Since $W(A - \alpha) = [\lambda_1 - \alpha, \lambda_2 - \alpha]$, then $|\lambda_2 - \alpha| = \|A - \alpha\|$. On the other hand, $\lambda_2 - \alpha \in \sigma(A - \alpha)$ and $A - \alpha$ is normaloid, then by Lemma 2.5,

$\lambda_2 - \alpha \in W_0(A - \alpha)$. Therefore, $\lambda_2 - \alpha \in W_0(A) \cap W_0(A - \alpha)$. This contradicts [7, Lemma 4], consequently $W_0(A) = \lambda_2$. Then, we have

$$W_0(A) = \begin{cases} [\lambda_1, \lambda_2]\,; & |\lambda_1| = |\lambda_2|, \\ \{\lambda_2\}; & |\lambda_1| < |\lambda_2|. \end{cases}$$

Example 2.7. Let $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator. If $A = \lambda I$ with λ is a real number, then $\overline{W(A)} = W_0(A) = \{\lambda\}$ and $c_A = \lambda$. Therefore, let A be non scalar with $\overline{W(A)} = [\alpha, \beta]$ where α and β are two real numbers with $\alpha < \beta$. By an argument similar to one in Example 2.6, we get

$$W_0(A) = \begin{cases} [\alpha, \beta]\,; & |\alpha| = |\beta|, \\ \{\alpha\}; & |\alpha| > |\beta|, \\ \{\beta\}; & |\alpha| < |\beta|. \end{cases}$$

If ever we have, for example, $W(A) = [\alpha, \beta)$ with $\|A\| = \beta$ (β must be positive, otherewise $\|A\| = |\alpha|$), then $\beta \in W_0(A)$. Indeed, since $\beta \in \overline{W(A)}$, there is a sequence of unit vectors $x_n \in \mathcal{H}$ such that $\beta = \lim_n \langle Ax_n, x_n\rangle$ and the Cauchy-Schwarz inequality implies that $\lim_n \|Ax_n\| = \|A\|$. So, $\beta \in W_0(A)$.

The previous example shows that if A is a non scalar selfadjoint operator then, $c_A \in W_0(A)$ if and only if $W_0(A) = \overline{W(A)}$, and necessarily $c_A = 0$.

We end this section by showing the non-continuity in the sens of Hausdorff of the maximal numerical range. First, let us recall the definition of Hausdorff distance. We denote by $\mathcal{K}(\mathbb{C})$ the set of all compact subsets of $\mathbb{C}$.

Definition 2.8. *Given $K, S \in \mathcal{K}(\mathbb{C})$, the Hausdorff distance is defined by*

$$h(K, S) = max(e(K, S), e(S, K))$$

where

$$e(K, S) = \sup_{x \in K} \inf_{y \in S} |x - y|.$$

It is proved in [6, Lemme 9] that the map

$$\psi : \ \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{K}(\mathbb{C}), \ A \longmapsto \overline{W(A)}$$

is continuous in the sens of Hausdorff. Unfortunately, it is not the case for the maximal numerical range. Let us take advantage of Example 2.7 to show this.

Proposition 2.9. *The map*

$$\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{K}(\mathbb{C}), \; A \longmapsto W_0(A)$$

is non continuous in the sens of Hausdorff.

Proof. According to Exemple 2.7, let $A \in \mathcal{B}(\mathcal{H})$ be an operator with $W_0(A) = [-1, 1]$, then $W_0(A) = [-1, 1]$ and set $A_n = A + \dfrac{1}{n}$ ($n = 1, 2, \dots$), so $W_0(A_n) = \{1 + \dfrac{1}{n}\}$. It is clear that $\lim_n \|A_n - A\| = 0$, then (A_n) converges uniformly to A in $\mathcal{B}(\mathcal{H})$. However,

$$\begin{aligned} e(W_0(A), W_0(A_n)) &= \sup_{x \in W_0(A)} \inf_{y \in W_0(A_n)} |x - y| \\ &= \sup_{x \in W_0(A)} |x - 1 - \frac{1}{n}| \\ &= 2 + \frac{1}{n}. \end{aligned}$$

Since $h(\varphi(A_n), \varphi(A)) = h(W_0(A_n), W_0(A)) \geqslant e(W_0(A), W_0(A_n))$, it follows that $\lim_n h(\varphi(A_n), \varphi(A)) \geqslant 2$. Consequently, φ is non continuous in the sens of Hausdorff. $\square$

References

[1] M. Barraa and M. Boumazgour, A note on the orthogonality of bounded linear operators, *Functional Analysis, Approximation and Computation*, 4 (1) (2012), 65–70.

[2] F. F. Bonsall and J. Duncan, Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras, *London Mathematical Society Lecture Note Series 2 Cambridge University Press, London-New York,* (1971).

[3] F. F. Bonsall and J. Duncan, Numerical Ranges II, *London Mathematical Society Lecture Notes Series 10 Cambridge University Press, New York-London,* (1973).

[4] K. E. Gustafson and D. K. M. Rao, Numerical Range: The Field of Values of Linear Operators and Matrices, *New York, NY, USA.,* (1997).

[5] P. R. Halmos, A Hilbert Space Problem Book, *Van Nostrand, New York,* (1967).

[6] M. C. Kaadoud, Domaine numérique du produit et de la bimultiplication $M_{2,A,B}$, *Proc. Amer. Math. Soc.*, 132 (8) (2004), 2421–2428.

[7] J. G. Stampfli, The norm of derivation, *Pacific J. Math.*, 33 (1970), 737–747.

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