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New Upper Bounds on the Spectral Radius of Graphs

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Abstract. We employ the concept of 2-degree of the vertices and, more generally, the number of walks between two vertices to introduce new upper and lower bounds for the spectral radius and the smallest eigenvalue of a graph. We, further, show how upper and lower bounds for the spectral radius are better than previous bounds in some cases.

AMS Subject Classification: 05C50
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1. Introduction

Throughout the paper we will assume $G = (V(G), E(G))$ to be a simple connected graph where the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and where $|E(G)| = m$. For $v_i \in V(G)$, the degree of v_i and the average of the degrees of the vertices adjacent to v_i are denoted by d_i and m_i, respectively. The 2-*degree* of a vertex v_i is defined to be the sum of the degrees of the vertices adjacent to v_i, i.e. $d_i m_i$. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ be the minimum and the maximum degree of the vertices of G, respectively. A k-walk of G is a list $v_{i_1} v_{i_2} \cdots v_{i_k}$ of the vertices of G such that the vertex v_{i_j} is adjacent to $v_{i_{j-1}}$, $2 \leqslant j \leqslant k$. We define $w_k(i)$ to be the number of k-walks starting with $v_i \in V(G)$; As well, for every pair of vertices $v_i, v_j \in V(G)$, we write $w_k(i,j)$ for the number of k-walks starting with v_i and ending with v_j.

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The adjacency matrix of a graph G is denoted by $A = A(G)$. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. Note that as $A(G)$ is a real symmetric matrix, all the eigenvalues of G are real. The spectral radius (i.e. the largest eigenvalue) and the smallest eigenvalue of a real symmetric matrix X are denoted by $\lambda(X) = \lambda$ and $\rho(X) = \rho$, respectively. We use similar notation $\lambda(G) = \lambda$ and $\rho(G) = \rho$ for a given graph G. Approximating the parameters $\lambda(G)$ and $\rho(G)$ is an important problem and has attracted much attention recently; see for example [11, 12, 15]. Below, we state some known upper bounds for the parameter $\lambda(G)$.

The first four upper bounds on the spectral radius of a graph are in terms of δ and Δ and number of the vertices and edges:

$$\text{See [2, 3]:} \qquad \lambda \leqslant \sqrt{2m - \delta(n-1) + \Delta(\delta - 1)} \tag{1}$$

$$\text{See [8]:} \qquad \lambda \leqslant \sqrt{2m - n - \delta + 2} \tag{2}$$

$$\text{See [14]:} \qquad \lambda \leqslant \sqrt{2m - n + 1 - (\delta - 1)(n - 1 - \Delta)} \tag{3}$$

$$\text{See [6, 10, 16]:} \qquad \lambda \leqslant \frac{1}{2}\left[\delta - 1 + \sqrt{(\delta+1)^2 + 4(2m - \delta n)}\right] \tag{4}$$

Furthermore, using the degree and the 2-degree of the vertices and average degrees of the vertices adjacent to $v_i \in V(G)$, some other upper bounds have been introduced:

$$\text{See [4]:} \qquad \lambda \leqslant \max_{v_i \in V(G)} \{m_i\} \tag{5}$$

$$\text{See [4]:} \qquad \lambda \leqslant \max_{v_i \in V(G)} \sqrt{d_i m_i} \tag{6}$$

$$\text{See [1]:} \qquad \lambda \leqslant \max_{v_i v_j \in E(G)} \sqrt{d_i d_j} \tag{7}$$

$$\text{See [3]:} \qquad \lambda \leqslant \max_{v_i v_j \in E(G)} \sqrt{m_i m_j} \tag{8}$$

In addition, Feng et al. [5] presented the following upper bounds for the spectral radius:

$$\lambda \leqslant \max_{v_i \in V(G)} \sqrt{\frac{d_i^2 + d_i m_i}{2}}. \tag{9}$$

$$\lambda \leqslant \max_{v_i v_j \in E(G)} \frac{\sqrt{d_i(d_i + m_i) + d_j(d_j + m_j)}}{2}. \tag{10}$$

$$\lambda \leqslant \max_{v_i \in V(G)} \frac{d_i + \sqrt{d_i m_i}}{2} \tag{11}$$

$$\lambda \leqslant \max_{v_i v_j \in E(G)} \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4 m_i m_j}}{4}. \tag{12}$$

It has been shown [3] that the bound (8) is always better than the bound (5). But some of the bounds (1)-(12) are incomparable. In this paper, we present some new upper and lower bounds for the spectral radius of a graph in terms of the 2-degrees and k-walks. We, further, obtain some bounds for the smallest eigenvalues of graphs in terms of k-walks. We will, finally, present some examples of graphs to compare our new bounds for the spectral radius with some of the bounds mentioned above as well as a bound due to Kummar [9].

We, first, recall the following fundamental result due to Brauer [7] which will be used in the paper.

Theorem 1.1. *Let $B = [b_{ij}]$ be an $n \times n$ matrix with entries in the complex field $\mathbb{C}$. All the eigenvalues of B are located in the union of the $n(n-1)/2$ ovals of cassini*

$$\bigcup_{\substack{i,j=1 \\ i \neq j}}^{n} \{z \in \mathbb{C} \, : \, |z - b_{ii}| \, |z - b_{jj}| \leqslant R_i(B) R_j(B)\}$$

where $R_i(B) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |b_{ij}|$.

The following fact [13] will, also, be useful.

Lemma 1.2. *For every graph G, $\delta(G) \leqslant \lambda \leqslant \Delta(G)$, where λ is the largest eigenvalue of G.*

2. Main Results

In this section we present the main results of the paper. We start with obtaining a new upper bound for the spectral radius of a graph G using the 2-degrees of their vertices.

Theorem 2.1. For any graph G with $n \geqslant 2$ vertices, we have

$$\lambda^2 \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\,(d_i m_i - d_i)(d_j m_j - d_j)}\right]. \tag{13}$$

The equality holds if G is a regular graph.

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of G. We apply Theorem **??** for the matrix $B = [b_{ij}] = A^2$. Since $b_{ii} = d_i$ and $R_i(B) = d_i m_i - d_i$, for all $i = 1, \ldots, n$, we conclude that $\lambda(B) = \lambda^2(A) = \lambda^2$ is located in the union of $\frac{n(n-1)}{2}$ ovals of cassini

$$\begin{aligned}
&\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{C} \;:\; |z - b_{ii}|\,|z - b_{jj}| \leqslant R_i(B)R_j(B)\}\\
&= \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{C} \;:\; |(z - d_i)(z - d_j)| \leqslant (d_i m_i - d_i)(d_j m_j - d_j)\}\\
&= \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{R} \cup (\mathbb{C} - \mathbb{R}) \;:\; |(z - d_i)(z - d_j)| \leqslant (d_i m_i - d_i)(d_j m_j - d_j)\}\,.
\end{aligned}$$

Since A is a real symmetric matrix and, thus, λ is a real eigenvalue, this implies that λ^2 is located in the region

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{R} \ : \ |(z-d_i)(z-d_j)| \leqslant (d_i m_i - d_i)(d_j m_j - d_j)\}$$

$$= \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{R} \ : \ -(d_i m_i - d_i)(d_j m_j - d_j)$$

$$\leqslant (z-d_i)(z-d_j) \leqslant (d_i m_i - d_i)(d_j m_j - d_j)\}$$

$$\subseteq \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{R} \ : \ (z-d_i)(z-d_j) \leqslant (d_i m_i - d_i)(d_j m_j - d_j)\}$$

$$\subseteq \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \left\{ z \in \mathbb{R} \ : \ z \leqslant \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4(d_i m_i - d_i)(d_j m_j - d_j)}}{2} \right\}.$$

from which the desired bound is achieved.

If G is a regular graph, then $d_i = \Delta$ and $d_i m_i = \Delta^2$, for all $v_i \in V(G)$. Thus

$$\max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4(d_i m_i - d_i)(d_j m_j - d_j)}}{2} = \Delta^2,$$

which, noting Lemma 1.2, proves the second part of the theorem. □

Now, using a similar approach as in Theorem 2.1, we present a new upper bounds for the spectral radius a graph G in terms of the number of walks. In what follows we will make use of the notation

$$W_{ij} = w_{m+1}(i,j) \quad \textit{and} \quad W_i = \frac{w_{m+p}(i)}{w_p(i)},$$

for each $i, j = 1, \ldots, n$ and every integers $p \geqslant 1$, $m \geqslant 1$.

Theorem 2.2 *For any graph G with $n \geqslant 2$ vertices and every integer $m \geqslant 1$, we have*

$$\lambda^m \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[W_{ii} + W_{jj} + \sqrt{(W_{ii} - W_{jj})^2 + 4(W_i - W_{ii})(W_j - W_{jj})} \right].$$

The equality holds if G is the complete graph K_n.

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of G and D be the diagonal matrix with diagonal $(w_p(1), ..., w_p(n))$. We apply Theorem **??** for the matrix $B = [b_{ij}] = D^{-1}A^m D$. Since $b_{ii} = W_{ii}$ and

$$R_i(B) = \sum_{j=1}^{n} W_{ij}\frac{w_p(j)}{w_p(i)} - W_{ii} = \frac{w_{m+p}(i)}{w_p(i)} - W_{ii} = W_i - W_{ii},$$

for all $i = 1, \ldots, n$, we conclude that $\lambda(B) = \lambda^m(A) = \lambda^m$ is located in the union of $\frac{n(n-1)}{2}$ ovals of cassini

$$\begin{aligned}&\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{C} \;:\; |z - b_{ii}|\,|z - b_{jj}| \leqslant R_i(B)R_j(B)\}\\ &= \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{C} \;:\; |\,(z - W_{ii})(z - W_{jj})\,| \leqslant (W_i - W_{ii})(W_j - W_{jj})\}.\end{aligned}$$

Since λ^m is a real number, this implies that λ^m is located in the region

$$\begin{aligned}&\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{R} \;:\; |(z - W_{ii})(z - W_{jj})| \leqslant (W_i - W_{ii})(W_j - W_{jj})\}\\ &= \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{R} \;:\; -(W_i - W_{ii})(W_j - W_{jj})\\ &\qquad\qquad \leqslant (z - W_{ii})(z - W_{jj}) \leqslant (W_i - W_{ii})(W_j - W_{jj})\}\\ &\subseteq \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{R} \;:\; (z - W_{ii})(z - W_{jj}) \leqslant (W_i - W_{ii})(W_j - W_{jj})\}\\ &\subseteq \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \left\{z \in \mathbb{R} \;:\; z \leqslant \frac{W_{ii} + W_{jj} + \sqrt{(W_{ii} - W_{jj})^2 + 4(W_i - W_{ii})(W_j - W_{jj})}}{2}\right\},\end{aligned}$$

from which the desired bound is achieved.

Let $G = K_n$. Then $W_{ii} = W_{jj}$ and $W_i = (n-1)^m$, for all $v_i \in V(G)$. Thus

$$\max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[W_{ii} + W_{jj} + \sqrt{(W_{ii} - W_{jj})^2 + 4(W_i - W_{ii})(W_j - W_{jj})}\right] = (n-1)^m,$$

which, noting Lemma 1.2, proves the second part of the theorem. □

Note that Theorem 2.1 is a spacial case of Theorem 2.2 where $m = 2$ and $p = 1$. Also, letting $m = 1$ and $p = 1, 2, 3$ in Theorem 2.2, we obtain the following known inequalities, respectively:

$$\lambda \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \sqrt{d_i d_j},$$

$$\lambda \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \sqrt{m_i m_j},$$

$$\lambda \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \sqrt{\frac{w_4(i)}{d_i m_i}\frac{w_4(j)}{d_j m_j}}.$$

Furthermore, we obtain new upper bounds for the spectral radius of a graph by setting $m = 2$ and $p = 2, 3, 4, 5$ in Theorem 2.2:

$$\lambda^2 \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\,(\frac{w_4(i)}{d_i} - d_i)(\frac{w_4(j)}{d_j} - d_j)}\right], \quad (14)$$

$$\lambda^2 \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\,(\frac{w_5(i)}{d_i m_i} - d_i)(\frac{w_5(j)}{d_j m_j} - d_j)}\right], \quad (15)$$

$$\lambda^2 \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\,(\frac{w_6(i)}{w_4(i)} - d_i)(\frac{w_6(j)}{w_4(j)} - d_j)}\right], \quad (16)$$

$$\lambda^2 \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\,(\frac{w_7(i)}{w_5(i)} - d_i)(\frac{w_7(j)}{w_5(j)} - d_j)}\right]. \quad (17)$$

Corollary 2.3. *For any graph G with $n \geqslant 2$ vertices and every integer $p \geqslant 1$, we have*

$$\lambda^2 \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\,(\frac{w_{p+2}(i)}{w_p(i)} - d_i)(\frac{w_{p+2}(j)}{w_p(j)} - d_j)}\right].$$

The equality holds if G is a regular graph.

Proof. The bound follows from Theorem 2.2 by letting $m = 2$. If G is a regular graph, then $d_i = \Delta$ and $\dfrac{w_{2+p}(i)}{w_p(i)} = \Delta^2$, for all $v_i \in V(G)$. Thus

$$\max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4(\frac{w_{2+p}(i)}{w_p(i)} - d_i)(\frac{w_{2+p}(j)}{w_p(j)} - d_j)}}{2} = \Delta^2,$$

which, noting Lemma 1.2, proves the second part of the theorem. □

Furthermore, noting that if G is a bipartite graph then $W_{ii} = w_{m+1}(i,i) = 0$, for each $i = 1, ..., n$, and for every odd integer $m \geqslant 1$, we obtain the following corollary of Theorem 2.2.

Corollary 2.4. *For any bipartite graph G with $n \geqslant 2$ vertices and every odd integer $m \geqslant 1$, we have*

$$\lambda^m \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \sqrt{W_i W_j}.$$

Also, taking $m = 1$ and $p = 1$, in Corollary 2.4, we obtain

$$\lambda \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \sqrt{d_i d_j}.$$

We now provide an example of some graphs for which some of our bounds are better than some of the bounds (1)-(12).

Example 2.5. Consider the graphs G, H and K in Figure 1. The value of each of the upper bounds (1)-(12) as well as our bounds are listed in Table 1. We observe that our bounds are the best among all known upper bounds for the graphs G and K, and that bound (4) is the best for the graph H. Thus, these upper bounds are not comparable.

Assuming $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a sequence of positive numbers, we define

$$W_i^\alpha = \sum_{j=1}^{n} W_{ij} \frac{\alpha_j}{\alpha_i}, \quad i = 1, \ldots, n.$$

One can generalize Theorem 2.2 and provide a stronger bound in terms of W_i^α as follows. The proof is similar to that of Theorem 2.2, where $w_p(i)$ is replaced by α_i.

Theorem 2.6. *Let G be a graph with $n \geqslant 2$. For every integer $m \geqslant 1$ and every sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ of positive numbers, we have*

$$\lambda^m \leqslant \max_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[W_{ii} + W_{jj} + \sqrt{(W_{ii} - W_{jj})^2 + 4({W_i}^\alpha - W_{ii})({W_j}^\alpha - W_{jj})}\right].$$

Now we obtain a lower bound for the spectral radius of a graph in terms of the number of its walks.

Theorem 2.7. *For any graph G with $n \geqslant 2$ vertices and every integer $m \geqslant 1$, we have*

$$\lambda^m \geqslant \max_{1\leqslant j<i\leqslant n} \frac{1}{2}\left[W_{ii}+W_{jj}+\sqrt{(W_{ii}-W_{jj})^2+4\,W_{ij}^2}\right]. \tag{18}$$

Proof. The largest eigenvalue of A^m is greater than or equal to the largest eigenvalue of any principal submatrix of A^m. Any principal submarix of order two of A^m is of the form

$$\begin{pmatrix} W_{ii} & W_{ij} \\ W_{ji} & W_{jj} \end{pmatrix}.$$

The m-th root of its largest eigenvalue is equal to

$$\left(\frac{W_{ii}+W_{ij}+\sqrt{(W_{ii}-W_{jj})^2+4W_{ij}^2}}{2}\right)^{\frac{1}{m}},$$

from which the inequality follows. $\square$

Taking $m = 2$ in Theorem 2.7, we obtain the following inequality which has been proved by Kummar [9].

$$\lambda^2 \geqslant \max_{1\leqslant j<i\leqslant n} \frac{d_i+d_j+\sqrt{(d_i-d_j)^2+4w_3^2(i,j)}}{2}.$$

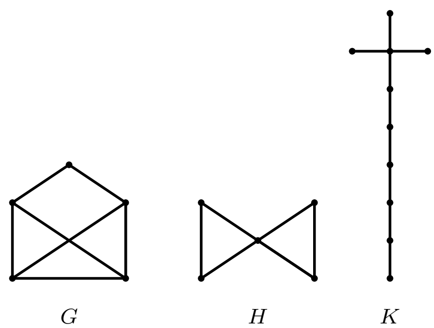

Figure 1. The graphs G , H , K

Table 1: Comparison of the upper bounds

inequality	graph G	graph H	graph K
(1)	3	2/8284	3
(2)	3	2/6457	3
(3)	3	2/8284	3
(4)	3	2/5615	3
(5)	3	3	4
(6)	3	2/8284	2/45
(7)	3	2/8284	2/83
(8)	3	3	2/45
(9)	3	3/3166	3/24
(10)	3	2/9154	2/78
(11)	3	3/4142	3/12
(12)	3	2/8228	2/60
(13)	3	2/6689	2/289
(14)	2/886751	2/6457	2/236
(15)	2/87228	2/5964	2/184
(16)	2/8675	3/5911	2/190
(17)	2/8656	2/5739	2/161
real value	2/8558	2/5616	2/119

Example 2.8. Consider the graph G in Figure 2. The numerical values of the lower bound (18), for some values of m for the graph G are given in Table 2. We observe that the new bound (18), when $m = 8$, is better than when $m = 2$, which is the bound in [9].

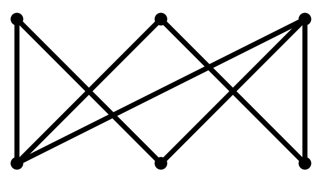

Figure 2. Graph G in Example 2.8.

We, next, turn our attention to the smallest eigenvalue of graphs and calculate, in the next theorems, several lower bounds for the smallest eigenvalue of a graph. We use a similar approach as in the proof of Theorem 2.2 to prove the following theorems.

Table 2: Table in Example 2.8.

m	2	3	4	5	6	7	8	
inequality (18)	2.13	1.91	2.25	2.10	2.30	2.18	2.32	max=2.32

Theorem 2.9. *For any graph G with $n \geqslant 2$ vertices and every odd integer $m \geqslant 1$, we have*

$$\rho^m \geqslant \min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \frac{1}{2}\left[W_{ii} + W_{jj} - \sqrt{(W_{ii}-W_{jj})^2 + 4(W_i - W_{ii})(W_j - W_{jj})}\right].$$

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of G and D be the diagonal matrix with diagonal $(w_p(1), ..., w_p(n))$. We apply Theorem **??** for the matrix $B = [b_{ij}] = D^{-1}A^m D$. Since $b_{ii} = W_{ii}$ and

$$R_i(B) = \sum_{j=1}^{n} W_{ij}\frac{w_p(j)}{w_p(i)} - W_{ii} = \frac{w_{m+p}(i)}{w_p(i)} - W_{ii} = W_i - W_{ii},$$

for all $i = 1, \ldots, n$, we conclude that $\rho(B) = \rho^m(A) = \rho^m$ is located in the union of $\frac{n(n-1)}{2}$ ovals of cassini

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{C} \;:\; |\,(z - W_{ii})(z - W_{jj})\,| \leqslant (W_i - W_{ii})(W_j - W_{jj})\}.$$

Since ρ^m is a real eigenvalue, this implies that ρ^m is located in the region

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in \mathbb{R} \;:\; (z - W_{ii})(z - W_{jj}) \leqslant (W_i - W_{ii})(W_j - W_{jj})\}$$
$$\subseteq \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \left\{z \in \mathbb{R} \;:\; z \geqslant \frac{W_{ii} + W_{jj} - \sqrt{(W_{ii}-W_{jj})^2 + 4(W_i - W_{ii})(W_j - W_{jj})}}{2}\right\},$$

from which the desired bound is achieved. □

One can generalize Theorem 2.9 and provide a stronger bound in terms of W_i^{α} as follows. The proof is similar to that of Theorem 2.9, where $w_p(i)$ is replaced by α_i.

Theorem 2.10. *Let G be a graph with $n \geqslant 2$. For every odd integer $m \geqslant 1$ and every sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ of positive numbers, we have*

$$\rho^m \geqslant \min_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \frac{1}{2}\left[W_{ii} + W_{jj} - \sqrt{(W_{ii} - W_{jj})^2 + 4({W_i}^{\alpha} - W_{ii})({W_j}^{\alpha} - W_{jj})}\right].$$

We conclude the paper with some upper bounds for $\rho = \rho(G)$ in terms of the number of walks of G.

Theorem 2.11. *For any graph G with $n \geqslant 2$ vertices and every odd integer $m \geqslant 1$, we have*

$$\rho^m \leqslant \min_{1 \leqslant j < i \leqslant n} \frac{1}{2}\left[W_{ii} + W_{jj} - \sqrt{(W_{ii} - W_{jj})^2 + 4W_{ij}^2}\right].$$

Proof. The proof is similar to that of Theorem 2.7; note that the smallest eigenvalue of A^m is less than or equal to the smallest eigenvalue of any 2×2 principal submatrix of A^m. □

The following is an immediate consequence of Theorem 2.11.

Corollary 2.12. *For any bipartite graph G with $n \geqslant 2$ vertices and every odd integer $m \geqslant 1$, we have*

$$\rho^m \leqslant \min_{1 \leqslant j < i \leqslant n} W_{ij}.$$

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