Journal of Mathematical Extension
Vol. 8, No. 1, (2014), 1-10

Weakly Completely Continuous Elements of the Banach Algebra LUC$(G)^*$

M. J. Mehdipour
Shiraz University of Technology

Abstract. In this paper, we study weakly compact left multipliers on the Banach algebra LUC$(G)^*$. We show that G is compact if and only if there exists a non-zero weakly compact left multipliers on LUC$(G)^*$. We also investigate the relation between positive left weakly completely continuous elements of the Banach algebras LUC$(G)^*$ and $L^\infty(G)^*$. Finally, we prove that G is finite if and only if there exists a non-zero multiplicative linear functional μ on LUC(G) such that μ is a left weakly completely continuous elements of LUC$(G)^*$.

AMS Subject Classification: 43A15; 47B07; 47B48
Keywords and Phrases: Locally compact group, multiplier, weakly compact operator

1. Introduction

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure λ. Let $L^1(G)$ be the group algebra of G defined as in [5] equipped with the convolution product $*$ and the norm $\|.\|_1$. Let $L^\infty(G)$ be the usual Lebesgue space as defined in [5] equipped with the essential supermum norm $\|.\|_\infty$. Then $L^\infty(G)$ is the dual of $L^1(G)$. We recall that the first dual $L^\infty(G)^*$ is a Banach algebra with the *first Arens product* "·" defined by $\langle F, H, f\rangle = \langle F, Hf\rangle$, where

$$\langle Hf, \phi\rangle = \langle H, f\phi\rangle, \qquad \text{and} \qquad \langle f\phi, \psi\rangle = \langle f, \phi * \psi\rangle,$$

for all $F, H \in L^\infty(G)^*$, $f \in L^\infty(G)$ and $\phi, \psi \in L^1(G)$.

Received: September 2013; Accepted: December 2013

Let $C(G)$ be the space of all bounded continuous complex-valued functions on G and $C_0(G)$ be the space of all continuous functions on G vanishing at infinity. The space of left uniform continuous function on G, denoted by $\mathrm{LUC}(G)$, is the set of all bounded continuous complex-valued functions f on G for which the mapping

$$x \mapsto \delta_x * f,$$

from G into $C(G)$ is norm continuous, where δ_x denotes the Dirac measure at x. The Banach space $\mathrm{LUC}(G)$ is a left introverted subspace of $L^\infty(G)$; that is, for each $\nu \in \mathrm{LUC}(G)^*$ and $f \in \mathrm{LUC}(G)$, the function νf defined by

$$\langle \nu f, x\rangle = \langle \nu, \delta_{x^{-1}} * f\rangle, \qquad (x \in G),$$

is also an element in $\mathrm{LUC}(G)$. This lets us to endow $\mathrm{LUC}(G)^*$ with the *first Arens product* "$\circ$" defined by

$$\langle \mu \circ \nu, f\rangle = \langle \mu, \nu f\rangle,$$

for all $\mu, \nu \in \mathrm{LUC}(G)^*$, $f \in \mathrm{LUC}(G)$. Then $\mathrm{LUC}(G)^*$ with this product is a Banach algebra.

Let π denote the natural continuous operator that associates to any functional in $L^\infty(G)^*$ its restriction to $\mathrm{LUC}(G)$. It is easy to see that π is a homomorphism and $\pi|_{E\cdot L^\infty(G)^*}$ is an isometric isomorphism from $E \cdot L^\infty(G)^*$ onto $\mathrm{LUC}(G)^*$ for all $E \in \Lambda(L^\infty(G)^*)$, the set of all mixed identities E with norm one in $L^\infty(G)^*$; that is,

$$\phi \cdot E = E \cdot \phi = \phi,$$

for all $\phi \in L^1(G)$. Also, observe that $\pi|_{L^1(G)}$ is identity on $L^1(G)$. Note that the group algebra $L^1(G)$ can be embedded into $\mathrm{LUC}(G)^*$ via

$$\langle \phi, f\rangle := \int_G \phi(x)\, f(x)\, d\lambda(x), \quad (\phi \in L^1(G), f \in \mathrm{LUC}(G)).$$

Let $\mathcal{A}$ be a Banach algebra; a bounded operator $T : \mathcal{A} \to \mathcal{A}$ is called *left multiplier* if $T(ab) = T(a)b$ for all $a, b \in \mathcal{A}$. For any $a \in \mathcal{A}$, the left multiplier $b \mapsto ab$ on $\mathcal{A}$ is denoted by λ_a; also a is said to be a *left* (*weakly*) *completely continuous element of* $\mathcal{A}$ if λ_a is a (weakly) compact operator on $\mathcal{A}$.

Sakai [10] has shown that if G is a locally compact non-compact group, then zero is the only left weakly completely continuous element of $L^1(G)$. Akemann [1] has proved that if G is compact, then any $\phi \in L^1(G)$ is a left weakly completely continuous element of $L^1(G)$. He also has characterized weakly compact left multiplier on $L^1(G)$. In fact, he has shown that any weakly compact left multiplier on $L^1(G)$ is of the form λ_ϕ for some $\phi \in L^1(G)$. Weakly compact left multipliers on the Banach algebra $L^\infty(G)^*$ of a locally compact group G have been studied by Ghahramani and Lau in [3,4]. In the same papers, they have obtained some results on the question of existence of non-zero weakly compact left multipliers on $L^\infty(G)^*$. Losert [8] among other things, has proved that if G is non-compact, then there is no non-zero weakly compact left multipliers on $L^\infty(G)^*$.

In this paper we study weakly compact left multipliers on the Banach algebra $\mathrm{LUC}(G)^*$ of a locally compact group G.

In Section 2, we show that G is compact if and only if there exists a non-zero weakly compact left multipliers on $\mathrm{LUC}(G)^*$. In Section 3, we investigate the relation between positive left weakly completely continuous elements of the Banach algebras $\mathrm{LUC}(G)^*$ and $L^\infty(G)^*$. We show that F is a positive left weakly completely continuous elements of $L^\infty(G)^*$ if and only if $F \in L^1(G)$ and it is a positive left weakly completely continuous elements of $\mathrm{LUC}(G)^*$. Finally, we prove that G is finite if and only if there exists a non-zero multiplicative linear functional μ on $\mathrm{LUC}(G)$ such that μ is a left weakly completely continuous elements of $\mathrm{LUC}(G)^*$.

2. The Existence of Weakly Completely Continuous Elements

Before we give the main result of this section, let us remark that any left multiplier T on $\mathrm{LUC}(G)^*$ is of the form λ_μ for some $\mu \in \mathrm{LUC}(G)^*$; indeed, $T = \lambda_{T(\delta_e)}$, where e denotes the identity element of G.

Theorem 2.1. *Let G be a locally compact group and $\mu \in LUC(G)^*$. Then μ is a non-zero left weakly completely continuous element of $LUC(G)^*$ if and only if G is compact and $\mu \in L^1(G)$.*

Proof. Let μ be a non-zero left weakly completely continuous element of $\mathrm{LUC}(G)^*$. Choose $E \in \Lambda(L^\infty(G)^*)$. Then there exists $F \in L^\infty(G)^*$ such that $\mu = \pi(E \cdot F)$. If $\pi_0 = \pi|_{E \cdot L^\infty(G)^*}$, then

$$\lambda_{E \cdot F} = \pi_0^{-1} \lambda_\mu \pi,$$

on $L^\infty(G)^*$. Hence $E \cdot F$ is a non-zero left weakly completely continuous element of $L^\infty(G)^*$. Thus G is compact and so $L^1(G)$ is an ideal in $L^\infty(G)^*$. Thus $\lambda_{E \cdot F}|_{L^1}(G) : L^1(G) \to L^1(G)$ is a weakly compact left multiplier. Hence there exists $\phi \in L^1(G)$ such that $\lambda_{E \cdot F} = \lambda_\phi$ on $L^1(G)$. Set $r := E \cdot F - \phi$. For every $\psi \in L^1(G)$ and $f \in C(G)$, we have

$$\langle r, \psi f \rangle = \langle (E \cdot F - \phi) \cdot \psi, f \rangle = 0.$$

From this and the fact that $L^1(G)C(G) = C(G)$, we see that $\langle r, g \rangle = 0$ for all $g \in C(G)$. Therefore, $\pi(r) = 0$; see Theorem 2.3 of [7]. We thus have

$$\mu = \pi(E \cdot F) = \pi(\phi) + \pi(r) = \phi \in L^1(G).$$

Conversely, let G be compact and $\mu \in L^1(G)$. Then μ is a left weakly completely continuous element of $L^1(G)$; see Theorem 4 of [1]. So, μ is a left weakly completely continuous element of $\mathrm{LUC}(G)^*$ by [2]. □

In the following, we give some corollaries of this theorem.

Corollary 2.2. *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) *G is compact.*
(b) *$LUC(G)^*$ has a non-zero positive left completely continuous element.*
(c) *$LUC(G)^*$ has a non-zero left completely continuous element.*
(d) *$LUC(G)^*$ has a non-zero left weakly completely continuous element.*

As an immediate consequence of Theorem 2.1 and Corollary 2.2, we have the following result.

Corollary 2.3. *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) *G is compact.*
(b) *Any $\phi \in L^1(G)$ is a left weakly completely continuous element of $LUC(G)^*$.*
(c) *$LUC(G)^*$ has a non-zero positive left weakly completely continuous element in $L^1(G)$.*
(d)*$LUC(G)^*$ has a non-zero left weakly completely continuous element in $L^1(G)$.*
(e) *$LUC(G)^*$ has a non-zero left weakly completely continuous element.*

In the following, for $C_0(G) \subseteq X \subseteq L^\infty(G)$, let

$$C_0(G)^{\perp_X} = \{r \in X^* : \langle r, f\rangle = 0 \quad \text{for all} \quad f \in C_0(G)\}.$$

As a consequence of Theorem 2.1, we present the next result.

Corollary 2.4. *Let G be a locally compact group and $r \in C_0(G)^{\perp_{LUC(G)}}$. If r is a left weakly completely continuous element of $LUC(G)^*$, then $r = 0$.*
Akemann [1] proved if $\mu \in L^1(G)$ is a left weakly completely continuous element, then it is a left completely continuous element. In this case, Ghahramani [2] showed that μ is a left weakly completely continuous element of $M(G)$. These results together with Theorem 2.1 prove the following corollary.

Corollary 2.5. *Let G be a locally compact group and $\mu \in LUC(G)^*$. Then μ is a left weakly completely continuous element of $LUC(G)^*$ if and only if μ is a left completely continuous element of $LUC(G)^*$.*

Let I be a subset of $LUC(G)^*$. The left annihilator of I in $LUC(G)^*$ is denoted by $lan(I)$ and is defined by

$$\text{lan}(I) = \{\mu \in \text{LUC}(G)^* : \mu \circ I = \{0\}\ \}.$$

Proposition 2.6. *Let G be a locally compact group and I be a closed right ideal in $LUC(G)^*$ such that $lan(I) = \{0\}$. Then G is compact if and only if there exists a non-zero weakly compact left multiplier on I.*

Proof. Let T be a non-zero weakly compact left multiplier on I. Then there exists $\xi \in I$ such that $T(\xi) \notin \operatorname{lan}(I)$. Hence $T(\xi \circ \zeta) \neq 0$ for some $\zeta \in I$. For every $\mu \in \mathrm{LUC}(G)^*$, we have

$$T(\xi \circ \zeta) \circ \mu = T(\xi) \circ \zeta \circ \mu = T(\xi \circ \zeta \circ \mu).$$

Therefore,

$$T\{\xi \circ \zeta \circ \mu : \mu \in \mathrm{LUC}(G)^*, \|\mu\| \leqslant 1\} \subseteq T\{\iota : \iota \in I, \|\iota\| \leqslant \|\xi\| \, \|\zeta\|\}.$$

This shows that $\lambda_{T(\xi \circ \zeta)}$ is a non-zero weakly compact left multiplier on $\mathrm{LUC}(G)^*$. Therefore, G is compact.
Conversely, let G be compact. Then $\mathrm{LUC}(G)^*$ has a non-zero left weakly completely continuous element, say μ. Since $\operatorname{lan}(I) = \{0\}$, we have $\mu \notin \operatorname{lan}(I)$. Hence λ_μ is a non-zero weakly compact left multiplier on I. $\square$

3. Positive Weakly Completely Continuous Elements

In this section, we study positive weakly completely continuous elements of the Banach algebras $\mathrm{LUC}(G)^*$ and $L^\infty(G)^*$. The main result of this section is the following.

Theorem 3.1. *Let G be a locally compact group and $F \in L^\infty(G)^*$. Then the following assertions are equivalent.*

(a) *F is a positive left completely continuous element of $L^\infty(G)^*$.*
(b) *F is a positive left weakly completely continuous element of $L^\infty(G)^*$.*
(c) *$F \in L^1(G)$ and F is a positive left weakly completely continuous element of $LUC(G)^*$.*
(d) *$F \in L^1(G)$ and F is a positive left completely continuous element of $LUC(G)^*$.*

Proof. The implication (a)$\Rightarrow$(b) is clear. By Corollary 2.5, (c)$\Rightarrow$(d). Now, if (d) holds, then F is a positive left completely continuous element of $L^1(G)$. The proof of Theorem 2.1 implies that F is a positive left completely continuous element of $L^\infty(G)^*$. Hence (d)$\Rightarrow$(a). To complete the proof, let F be a positive left weakly completely continuous element of $L^\infty(G)^*$. Then $\pi(F)$ is a left weakly completely continuous element of $\mathrm{LUC}(G)^*$. Choose $E \in \Lambda(L^\infty(G)^*)$. From Theorem 2.1 and the fact that

$$\pi(E \cdot F) = \pi(F),$$

we see that $\pi(E \cdot F) \in L^1(G)$. Since $\pi : E \cdot L^\infty(G)^* \to \mathrm{LUC}(G)^*$ is an isometry and π is identity on $L^1(G)$, we have $E \cdot F \in L^1(G)$. Set $r := F - E \cdot F$. We show that $r = 0$. Suppose r is non-zero. Let g be a continuous complex-valued functions on G with compact support C such that $\|g\| \leqslant 1$ and

$$|\langle E \cdot F, g\rangle| \geqslant \|E \cdot F\| - (5/12)\|r\|.$$

Choose an element f in the unite ball $L^\infty(G)$ such that

$$|\langle r, f\rangle| \geqslant (23/24)\|r\|.$$

Let V be an open set with compact closure for which $C \subseteq V$. Let h be a continuous complex-valued functions on G such that $0 \leqslant h(x) \leqslant 1$ for all $x \in G$, $h(x) = 1$ for all $x \in C$ and $h(x) = 0$ for all $x \notin V$. Define the complex-valued function $\jmath$ on G by

$$\jmath(x) := f(x) - h(x)g(x) + g(x),$$

for all $x \in G$. There exists a complex number η such that $\|\eta(f - hg) + g\| \leqslant 1$ and

$$|\langle E \cdot F, \eta(f - hg) + g\rangle| = |\langle E \cdot F, f - hg\rangle| + |\langle E \cdot F, g\rangle|.$$

Since $\langle r, k\rangle = 0$ for all $k \in C(G)$, it follows that $\langle r, \jmath\rangle = \langle r, f\rangle$. Thus

$$\begin{aligned} |\langle F, \jmath\rangle| &\geqslant |\langle r, \jmath\rangle| + |\langle E \cdot F, g\rangle| - |\langle E \cdot F, f - hg\rangle| \\ &\geqslant |\langle r, f\rangle| + 2|\langle E \cdot F, g\rangle| - \|E \cdot F\| \\ &\geqslant (1/8)\|r\| + \|E \cdot F\|. \end{aligned}$$

Note that $\|\jmath\| \leqslant 1$. Therefore,

$$\|F\| \geqslant (1/8)\|r\| + \|E \cdot F\|.$$

Let χ_G be the characteristic function of G. Since F is positive and $\langle r, \chi_G\rangle = 0$, we have

$$\|F\| = \langle F, \chi_G\rangle = \langle E\cdot F, \chi_G\rangle = \|E\cdot F\|.$$

Thus $\|r\| = 0$ and so $r = 0$, a contradiction. Therefore,

$$F = E\cdot F \in L^1(G).$$

Now, if $F \neq 0$, then G is compact; see [8]. Therefore, F is a positive left weakly completely continuous element of $\mathrm{LUC}(G)^*$ by Theorem 2.1. That is, (b)$\Rightarrow$(c). $\square$

As an immediate consequence of this theorem, we have the following result.

Corollary 3.2. *Let G be a locally compact group and $r \in C_0(G)^{\perp_{L^\infty(G)}}$. If r is a positive left weakly completely continuous element of $L^\infty(G)^*$, then $r = 0$.*

Let X be a closed C^*−subalgebra of $L^\infty(G)$. We denote by $\Omega(X^*)$ the set of all non-zero multiplicative linear functionals on X^*.

Theorem 3.3. *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) *G is finite.*
(b) *$L^\infty(G)^*$ has a left weakly completely continuous element in $\Omega(L^\infty(G)^*)$.*
(c) *$LUC(G)^*$ has a left weakly completely continuous element in $\Omega(LUC(G)^*)$.*

Proof. It is trivial that (a) implies (b). Let $F \in \Omega(L^\infty(G)^*)$ be a left weakly completely continuous element of $L^\infty(G)^*$. Then F is a positive left weakly completely continuous element of $L^\infty(G)^*$. By Theorem 3.1, $F \in L^1(G)$ and it is a positive left weakly completely continuous element of $\mathrm{LUC}(G)^*$. It is clear that

$$F \in \Omega(\mathrm{LUC}(G)^*).$$

That is, (b) implies (c). Let $\mu \in \Omega(\mathrm{LUC}(G)^*)$ be a left weakly completely continuous element of $\mathrm{LUC}(G)^*$. Then G is compact and $\mu \in$

$L^1(G)$. Thus $\mu = \mu|_{C_0(G)}$. Hence $\mu = \delta_x$ for some $x \in G$; see [6]. So $\delta_x \in L^1(G)$; equivalently, G is discrete. Therefore, G is finite; that is (c) implies (a). □

We conclude the paper with the following result.

Proposition 3.4. *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) *G is finite.*
(b) *Any element $F \in L^\infty(G)^*$ is a left weakly completely continuous element of $L^\infty(G)^*$.*
(c) *Any positive element $F \in L^\infty(G)^*$ is a left weakly completely continuous element of $L^\infty(G)^*$.*
(d) *Any positive element $\mu \in LUC(G)^*$ is a left weakly completely continuous element of $LUC(G)^*$.*
(e) *Any element $\mu \in LUC(G)^*$ is a left weakly completely continuous element of $LUC(G)^*$.*

Proof. The implications (a)⇒(b)⇒(c) are clear. Let μ be a positive left weakly completely continuous element of LUC$(G)^*$. By Hahn-Banach theorem, there is positive element $F \in L^\infty(G)^*$ such that $\mu = \pi(F)$; see [9]. If (c) holds, then F is a positive left weakly completely continuous element of $L^\infty(G)^*$. In view of Theorem 3.1, we have $F \in L^1(G)$. It follows that

$$\mu = \pi(F) = F \in L^1(G).$$

Therefore, μ is a positive left weakly completely continuous element of LUC$(G)^*$ by Theorem 2.1. That is, (c)⇒(d). To prove (d)⇒(e), we only need to note that if $\mu \in$ LUC$(G)^*$, then $\mu = \sum_{i=1}^{4} \alpha_i \mu_i$ for some positive elements μ_i of LUC$(G)^*$ and $\alpha_i \in \mathbb{C}$ $(i = 1, 2, 3, 4)$. Finally, by Theorem 3.3, (e)⇒(a). □

References

[1] C. A. Akemann, Some mapping properties of the group algebras of a compact group, *Pacific J. Math.*, 22 (1967), 1-8.

[2] F. Ghahramani, Compact elements of weighted group algebras, *Pacific J. Math.*, 113 (1984), 77-84.

[3] F. Ghahramani and A. T. Lau, Isomorphisms and multipliers on second dual algebras of Banach algebras, *Math. Proc. Cambridge Philos. Soc.*, 111 (1992), 161-168.

[4] F. Ghahramani and A. T. Lau, Multipliers and modulus on Banach algebras related to locally compact groups, *J. Funct. Anal.*, 150 (1997), 478-497.

[5] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, New-York, 1970.

[6] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New-York, 1975.

[7] A. T. Lau and J. Pym, Concerning the second dual of the group algebra of a locally compact group, *J. London Math. Soc.*, 41 (1990), 445-460.

[8] V. Losert, Weakly compact multipliers on group algebras, *J. Funct. Anal.*, 213 (2004), 466-472.

[9] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Boston, 1990.

[10] S. Sakai, Weakly compact operators on operator algebras, *Pacific J. Math.*, 14 (1964), 659-664.

Mohammad Javad Mehdipour
Department of Mathematics
Assistant Professor of Mathematics
Shiraz University of Technology
P. O. Box: 71555-313
Shiraz, Iran
E-mail: mehdipour@sutech.ac.ir