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Foundations on G-Type Domains

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Abstract. In this article by using G-type domains, we introduce strong G-type domains and locally countable quotient rings(lcqr). Morover, G-type ideals are classified. Finally some relations between prime ideals and G-type ideals in valuation rings have been investigated.

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1. Mathematical Notations

Definition 1.1. *A domain R with its quotient field Q is called a G-type domain if Q as a ring on R is countably generated (c.g.), i.e., there exists a countable multiplicative closed subset* ***(cmcs)*** *in R such that:*

$$Q = S^{-1}R = R[1/S]$$

Note: An Ordinal number of λ is called the Caliber of R. It is denoted by: "$C_a(R) = \lambda$". For a multiplicative closed subset M of R, which is the smallest multiplicative closed subset generated by S, if $M^{-1}R = Q$ then $|S| = \lambda$.[9]

Lemma 1.2. *A domain R is a* ***G-type domain*** *if and only if $C_a(R) \leqslant \aleph_0$.*

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Definition 1.3. *A prime ideal P of domain R is called a **G-type ideal** if the quotient ring R/P is a G-type domain.*

Theorem 1.4. [8] *Let P be a prime ideal of R, the following statements are equivalent:*
i)*P is a **G-type ideal** of R.*
ii)*There exists a **cmcs** set S in R such that P is maximal with respect to having the empty intersection with S.*
iii)*There is either only a countable number of prime ideals in R/P or any uncountable set of prime ideals properly containing P , say F, can be written in the form $F = \bigcup_{n\in\Lambda} F_n$, where Λ is a subset of the natural numbers, P is properly contained in $\bigcap_{Q\in F_n} Q$ for each $n \in \Lambda$ and some of the F_n are uncountable and $F_n = \{Q \in F : s_n \in Q\} \neq \emptyset$.*

Definition 1.5. [9] *Let R be a commutative domain and Q its quotient field. R is called a **strong G-type domain** if every overring $R^{'}$ of R is the form of $R[1/t_1, 1/t_2, 1/t_3, ...]$, for some nonzero* elements $t_i \in R$,
in other words,

$$R' = R[1/S] = S^{-1}R$$

where $S =< \{t_1, t_2, ..., t_n, ...\} >$ is a ***cmcs*** of R.

Definition 1.6. *A quotient ring R' of a domain R is called a **countable quotient ring (cqr)** if $R' = R_S$, for a **cmcs**, $S =< \{t_1, t_2, t_3, ...\} >$ where $\quad t_i \neq 0, \quad \forall i \in I$.*

Example 1.7. The Integral domain of $\mathcal{Z}$ is a trivial sample for Definition 1.4.

Definition 1.8. *R is called a **locally countable quotient ring (lcqr)** if for every prime ideal P of R, the localization of R_P is a **cqr** of R.*

Lemma 1.9. *Every strong **G-type domain** is a **lcqr**.*

Proof. For each arbitrary prime ideal P of R, R_P is also an overring of R, so by the property of strong **G-type domains** and by Definition 1.4, there exists a cmcs, $S =< \{t_1, t_2, ..., t_n, ...\} >$ of R, such that P with respect to inclusion satisfies the property of $P \cap S = \emptyset$.

Therefore we have:

$$R_P = S^{-1}R(= R_S). \quad \square$$

Lemma 1.10. *Every **lcqr** is a **G-type domain**.*

Proof. Since R is a domain, so $\{0\}$ is also prime ideal of R. Therefore, $R_{\{0\}}$ is a ***cqr*** i.e., there exists a ***cmcs*** set S of R such that $R_{\{0\}} = S^{-1}R$. Since R is a domain; therefore, $R_{\{0\}}$ is its quotient field "say, K". So

$$K = R_{\{0\}} = S^{-1}R.$$

Hence R is a **G-type domain**. $\square$

2. Properties of *LCQR* Domains

In this section, some key results have been drawn for the ***lcqr*** domains which are defined in §1. The importance of these results lies in the countability of the number of maximal ideals in these domains.

Theorem 2.1. *Let P be a prime ideal in a domain of R and S be a **mcs** (set of nonzero elements of R) such that $S \cap P = \emptyset$, then the following statements are equivalent:*
i) $R_P = R_S(= S^{-1}R)$ *is a **cqr** of R.*
ii) $S \cap (b) \neq \emptyset$ *for every element b $inR\backslash P$.*
iii) *If Q is a prime ideal not contained in P, then $S \cap Q \neq \emptyset$.*

Proof. $(i) \to (iii)$: Let $R_P = R_S$ and Q be a prime ideal of R which isn't contained in P, and $q \in Q\backslash P$, then $1/q \in R_P$. Therefore, $1/q = a/s$ for some $a \in R$ and $b \in S$. Hence $s = aq \in Q \cap S$, $S \cap Q \neq \emptyset$.
$(iii) \to (ii)$: For $b \notin P$ and $S \cap (b) = \emptyset$ it reaches contradiction. By Cohen's theorem, there exists a prime ideal Q containing (b) such that $S \cap Q = \emptyset$, but $Q \nsubseteq P$, this is a contradiction by (iii).
$(ii) \to (i)$: By the hypothesis since $S \cap P = \emptyset$, then $S \subseteq R\backslash P$ i.e., the quotient ring $R_S \subseteq R_P$. Now let $x = a/b \in R_P$ with $a, b \in R$ and $b \notin P$, then by (ii) it seems that $S \cap (b) \neq \emptyset$. Therefore for some $r \in R$, we can put $s = br \in S$. Hence, $x = a/b = ar/br \in R_S$. $\square$

Corollary 2.2. *A domain R with a countable number of prime ideals satisfies in the following properties:*
i) *R is a G-type domain.*
ii) R is an ***lcqr***

Proof. It is obvious that R is a G-type domain, [8,Corollary 1.3].
Now let $\{P_1, P_2, ...\}$ be the set of all prime ideals in R. Therefore, for each P in $Spec(R)$ it must be $P = P_i$, for some $i \in I$.
Now we are setting $F = \{P_j : P_j \not\subseteq P\}$. If $F = \emptyset$, then this means that R is local and P is its unique maximal ideal, Therefore, $S = R \backslash P$ consists of all the units of R, hence

$$R = R_P = S^{-1}R = R_1.$$

Thus we may assume that $F \neq \emptyset$ and hence for each $P_j \not\subseteq P$, take $a_j \in P_j \backslash P$ and put

$$T =< \{a_j : j \in J\} > .$$

Clearly T is a ***cmcs*** in R. Obviously, for each $Q \not\subseteq P$, we have $T \cap Q \neq \emptyset$. Therefore, by parts (i) and (iii) of Theorem 2.1 $R_P = R_T$, hence the proof is completed. □

Corollary 2.3. *Let R be a domain and R_P be a **cqr** of R, then P is a G-type ideal.*

Proof. R_P is a ***cqr***. Therefore, there exists a ***cmcs*** set S in R such that

$$R_P = R_S (= S^{-1}R).$$

Now by [8, Definition 4.1], S is satisfying in property of $S \cap P = \emptyset$, now by part (iii) of Theorem 2.1, since for each $Q \supset P$, it's been $S \cap Q \neq \emptyset$; hence P is maximal with respect to its property ($S \cap P = \emptyset$).
Therefore, by Theorem 2.1, P is a G-type ideal of R. □
The following is more immediate.

Corollary 2.4. *Every prime ideal in a **lcqr** domain is a G-type ideal.*

Proof. Let P be a prime ideal of R. By hypothesis R_P is a ***cqr*** of R. Hence, by Corollary 2.2, P is a G - type ideal of R. □

Corollary 2.5. *Let P be a prime ideal of a domain R and suppose the localization R_P of R is a* ***cqr****, then Y_P=$\{Q \subseteq P : Q \in Spec(R)\}$ is an intersection of open subsets of $Spec(R)$ with the Zariski topology.*

Proof. By our hypothesis, there exists a ***cmcs*** set S in R such that

$$R_P = R_S(= S^{-1}R).$$

Now by Theorem 2.1 for each prime ideal Q, which is not contained in P and $Q \cap S \neq \emptyset$ (S is countable), there exists $\{t_1, t_2, ...\} \subseteq S$ such that

$$Q \cap S = \{t_1, t_2, ...\}(= S_Q), \quad i.e. \quad Y_P^c = \bigcup_Q V(S_Q).$$

Therefore $Y_P = (\bigcup_Q V(S_Q))^c$ and then $Y_P = \bigcup_Q D(S_Q)$,
where $D(S_Q)$ is an open subset of $X = Spec(R)$ with the Zariski topology. □

Note. If R is a semilocal ring, i.e., let $M_1, M_2, ..., M_n$ be all maximal ideals of R, and for each i, $(1 \leqslant i \leqslant n)$, if $A_i = \prod_{j \neq i} M_j$, then there have been finite numbers of ideals $A_1, A_2, ..., A_n$ such that for all i,

$$A_i \nsubseteq M_i,$$

and for each prime ideal P of R, it must be satisfying $P \nsupseteq A_i$, for some $1 \leqslant i \leqslant n$.

Hence in extended position if R is a semilocal ring then there exists a finite number of ideals $A_1, A_2, ..., A_n$ "not necessary maximal" such that for each arbitrary prime ideal $P \in Spec(R)$, $P \nsupseteq A_i$ for some i.

Theorem 2.6. *Let R be a* ***lcqr*** *domain, then there exists only a finite number of maximal ideals $M_1, M_2, ..., M_n$ with the finite number of ideals $A_1, A_2, ..., A_n$ of R such that:*

$$M_i + A_i = R \quad , \quad i = 1, 2, ..., n$$

and for each prime ideal $P \in Spec(R)$ there exists some i, such that $A_i \nsubseteq P$.

In other words, for each maximal M, we have $M + A_i = R$ for some i.

Proof. Let $X = Spec(R)$ and $\{M_i\}_{i\in I}$ be the set of all maximal ideals in R. Then by Corollary 2.4, we have:

$$Y_{m_i} = \bigcap_{j\in F_i} G_j^i \quad , \quad i \in I$$

where each G_j^i is an open set in the Zariski topology.
Now we have:

$$X \subseteq \bigcup_{i\in I} G_j^i \quad \Longrightarrow \quad X \subseteq \bigcup_{i\in I} Y_{M_i} \subseteq \bigcup_{i\in I} G_j^i.$$

Since X is compact, therefore:

$$X = G_j^{i_1} \cup G_j^{i_2} \cup ... \cup G_j^{i_n} = D(A_{i_1}) \cup D(A_{i_2}) \cup ... \cup D(A_{i_n}),$$

where $G_j^{i_k} = D(A_{i_k})$ and A_{i_k} is an ideal of R.
Therefore we have:

$$M_{i_1} \in G_j^{i_1} = D(A_{i_1}),$$
$$M_{i_1} \nsupseteq A_{i_1},$$
$$M_{i_1} + A_{i_1} = R,$$
$$.$$
$$.$$
$$.$$
$$M_{i_n} + A_{i_n} = R.$$

Now by the last note for each $P \in X$, for some k, we have

$$P \in G_j^{i_k} = D(A_{i_k}) \text{ and } P \nsupseteq A_{i_k}$$

The final part is evident. □

Theorem 2.7. *In an **lcqr** domain, any descending chain of prime ideals is at most countable.*

Proof. If we suppose that $Q_1 \supset Q_2 \supset ... \supset Q_\omega \supset Q_{(\omega+1)} \supset Q_{(\alpha)} \supset ...$ is a strictly descending chain of uncountable number prime ideals of R, (ω is the first infinite ordinal and $\alpha < \omega_1$) and ω_1 is the first uncountable ordinal, then we seek a contradiction.
Now let P be an ideal with the following condition

$$P = \bigcap_{i \geqslant 1} Q_i.$$

Since P is also a prime ideal of R, so R_P is also ***cqr*** of R, then there exists a countable multiplicative closed subset **cmcs** $S \subseteq R$ such that $R_P = R_S$ and $P \cap S = \emptyset$, then by part (iii) of Theorem 2.1 for every $i \geqslant 1$ since $Q_i \nsubseteq P$ so we have

$$Q_i \cap S \neq \emptyset.$$

Now we may consider the following chain

$$S \cap Q_1 \supset S \cap Q_2 \supset ... \supset S \cap Q_\omega \supset S \cap Q_{(\omega+1)} \supset ... \,.$$

Since For each $i \geqslant 1$, $S \cap Q_i$ is a countable set, then we must have

$$S \cap Q_\beta = S \cap Q_i = S \cap Q_{i+1}, \quad \forall i \geqslant \beta.$$

For some $\beta < \omega_1$, we may take β to be the least ordinal with this property. Consequently, we have

$$\bigcap_{i \geqslant 1} (S \cap Q_i) = S \cap Q_\beta \neq \emptyset.$$

But it means that

$$(\bigcap_{i \geqslant 1} Q_i) \cap S = P \cap S = Q_\beta \cap S \neq \emptyset.$$

which is a contradiction. $\square$

Corollary 2.8. *Every quotient ring of an* ***lcqr*** *domain is also an* ***lcqr****.*

Proof. Let $R' = R_S$ be a quotient ring of an ***lcqr*** domain R with respect to a ***cmcs*** set S. It is obvious that by canonical homomorphism

$$\phi : R \longrightarrow S^{-1}R$$

with $\phi(r) = \frac{r}{1}$, every prime ideal $P^{'}$ of $R^{'}$ is the form of P^e which is the extension of prime ideal P of R such that $P \cap S = \emptyset$.
In addition, by [5], we have

$$R'_{P'} \simeq R_P = \frac{R_S}{P^e}.$$

Now since R is ***lcqr***, there exists an ***cmcs*** T of R such that

$$T \cap P = \emptyset.$$

So we have $R_P = R_T$. Therefore $R^{'}_{P'} \simeq R_P = R_T$, and hence the proof is completed. □

Lemma 2.9. *Let $\{S_i\}_{i \in I}$ be a countable chain of* ***cmcs*** *sets in R. If $S = \bigcup_{i \in I} S_i$, which is a* ***cmcs*** *in R , then*

$$\bigcap_{i \in I} R_{S_i} \subseteq R_S.$$

Proof. It is obvious that $S \neq \emptyset$.
Now let $0 \neq \alpha \in \bigcap_{i \in I} R_{S_i}$ be an arbitrary element, then

$$\alpha \in R_{S_i} \quad , \quad \forall i \in I.$$

So there exists $s_i \in S_i$ and $a_i \in R$ such that

$$\alpha = a_i / s_i \quad , \quad \forall i \in I.$$

Now since for all $i \in I$, $S_i \subseteq S$, therefore $s_i \in S$ for each $i \in I$, and so we have

$$\alpha \in R_S.$$

Hence $\bigcap_{i \in I} R_{S_i} \subseteq R_S$. □

Lemma 2.10. *Let $\{P_i\}_{i\in I}$ be a chain of prime ideals in R and suppose that for all $i \in I$, R_{P_i} is a* ***cqr*** *of R, then $R' = \bigcap_{i\in I} R_{P_i}$ is* ***cqr*** *of R.*

Proof. By Theorem 2.1, since for each $i \in I$, R_{P_i} is a ***cqr*** of R, then there exists a ***cmcs***, S_i in R such that P_i with respect to inclusion is satisfying $S_i \cap P_i = \emptyset$ and we have

$$R_{P_i} = S_i^{-1}R = R[\frac{1}{S_i^{-1}}](= R_{S_i}), \quad \forall i \in I.$$

Then $\bigcap_{i\in I} R_{P_i} = \bigcap_{i\in I} R_{S_i}$.
Now if we define $S = \bigcup_{i\in I} S_i$, then by Lemma 2.1, S is a ***cmcs*** in R. Since for each $i \in I$, $P_i = R\backslash S_i$ with respect to inclusion is maximal which is satisfying $P_i \cap S_i = \emptyset$ then $\cap P_i = \cap(R\backslash S_i) = R\backslash(\cup S_i) = R\backslash S$ and so $S = R\backslash \cap P_i$, where S is a ***cmcs*** in R.
Therefore, $R_S = \bigcap_{i\in I} R_{P_i} = \bigcap_{i\in I} R_{S_i}$. $\square$

Note. $Gold(v)$, is defined as the set of all G-ideals of V.

Theorem 2.11. *Let V be a valuation ring. Then, the following statements are equivalent:*

i) *V is a strong* ***G-type domain****.*
ii) *V is an* ***lcqr*** *domain.*
iii) *$Spec(V) = Gold(V)$.*
iv) *For every prime ideal P of V either V/P has only a countable number of prime ideals or the set of all prime ideals properly containing P, say F, can be written in the form $F = \bigcup_{n\in T} F_n$, where T is a subset of Natural number and each F_n is a well-ordered set. (Note: Clearly some F_n is uncountable).*

Proof. $(i) \to (ii)$: It's the proof of lemma (1.3).
$(ii) \to (iii)$: It's the corollary (2.3).
$(iii) \to (iv)$: The first part is the proof of $(2 \Longrightarrow 3)$ of Proposition 1.10 of [10]. For the second part, let $F = \bigcup_{n\in T} F_n$, and P be a prime ideal of R such that it's properly contained in $\bigcap_{Q\in F_n} Q$ for each $n \in T$, this means that $\bigcap_{Q\in F_n} Q$ equal to $Q' \neq 0$ is a prime ideal containing P.
Now if $F = \{Q \in Spec(V) : P \subset Q\}$, then

$$F_n = \{Q \in F : s_n \in Q\},$$

where $S =< \{s_1, s_2, ...\} >$ is a ***mcs*** set in V such that P is maximal with respect to having the empty intersection with S.
Consequently,

$$s_n \in \bigcap_{Q \in F_n} Q = Q^{'},$$

that is $Q^{'} \in F_n$, i.e., F_n is a well-ordered set.
$(iv) \rightarrow (i)$: The proof is the same as proof of $(3 \rightarrow 1)$ of Proposition 1.10 of [10]. □

Example 2.12. By the following, we have presented a domain which is an "***lcqr***" but not a "$StrongG - Typedomain$".
It is well-known that the integer Number of $\mathbb{Z}$ is a G-type domain but it is not a G-domain and each of its prime ideals "P" as the form of $< p >$, where p is a prime number of $\mathbb{Z}$. So for every prime ideal of $\mathbb{Z}$ say "P", there exists a countable multiplicative closed subset "S" of $\mathbb{Z}$ (which contains all prime elements of $\mathbb{Z}$ except "p") such that

$$\mathbb{Z}_{<p>} = \mathbb{Z}_S.$$

This means that $\mathbb{Z}$ is an "***lcqr***" domain.
Now if $\mathbb{Z}^{'}$ is an overring of $\mathbb{Z}$ contained in $\mathbb{Q}$, then $\mathbb{Z}^{'}$ can be expressed as follows

$$\mathbb{Z}^{'} = \mathbb{Z}[\frac{1}{p_1}, \frac{1}{p_2}, ..., \frac{1}{p_k}]$$

where k is a finite number.
It seems that there does not exist any prime ideal of $\mathbb{Z}$ which contains all prime elements of $\mathbb{Z}$ except the prime elements $p_1, p_2, ..., p_k$.
Therefore, $\mathbb{Z}$ can't be a strong G- type domain.
Now we are producing a **G-type domain** which is not an "***lcqr***".

Example 2.13. Let V be a valuation **G-type domain**; constructed as follows,
suppose that V is an ordered group generated by lexicographic product

of Hahn groups as: $\mathcal{H}_{i\in I\cup\omega}\mathfrak{F}_i$, where I is an uncountable set and for each $i \in I$ and $\omega > i$, $\mathfrak{F}_i$ is an ordered field of Real number, so V will be a **G-type domain** Now if we define:

$$\mathcal{H}_{i\in I\cup\omega}\mathfrak{F}_i/\mathcal{H}_{i\in I\setminus\omega}\mathfrak{F}_i \simeq \mathfrak{F}_\omega$$

where $\mathcal{H}_{i\in I\setminus\omega}\mathfrak{F}_i = P_\omega$, which is a prime ideal of V; therefore, it could be V_{P_ω} not only isn't an "***lcqr***" but also it is not a domain [11].

Definition 2.14. *A domain R has the countable overring property if every overring of R is a countably generated ring over R.*

Theorem 2.15. *If a domain R has the countable overring property, then R is* ***lcqr***

Proof. Let P be an arbitrary prime ideal of R and let $R_P = R[a_1/b_1, ...,$ $a_n/b_n, ...]$ with $a_i, b_i \in R$, we may write $a_i/b_i = r_i/t_i$ with $t_i \notin P$. If $T = \{t_i : i \in I\}$; clearly T is countable and it can generate an ***mcs*** set S in R such that it is maximal with respect to $S \cap P = \emptyset$. Now if q is any nonzero element of R such that it does not belong to P (i.e.,$0 \neq q \in R\setminus P$) then by Cohn's theorem there exists a prime ideal Q in $Spec(R)$ such that it contains (q); now if $S \cap Q = \emptyset$; then by maximality of P with respect to this property, then $Q \subseteq P$ (i.e.,$q \in P$) which is a contradiction. Therefore, $S \cap (q) \neq \emptyset$, and by part (iii) of Theorem 2.1 we have $R_P = R_S$, so R is ***lcqr***. □

The author suggests that further research in this direction which is likely to reveal additional properties of Noetherian **G-type domains** and thus may contribute to our understanding of how such structures may be defined on the underlying $G\lambda$-type domains , where λ is any regular cardinal.

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