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A Concise Formula for Two Dimensional Divided Differences with Multiple Knots

R. Roozegar*
Yazd University

A. A. Jafari
Yazd University

Abstract. The well-known formulas for divided differences in the case of one and two dimensional with multiple knots have played an important role in applied mathematics, particularly in numerical analysis, polynomial and lagrange interpolations. In this article we obtain a new formula for two dimensional divided differences in the case of multiple knots. This formula has a simpler form than the known formula given in literature. It is friendly using for computations and analysis.

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1. Introduction

The divided difference has appeared to be a powerful tool in certain areas of applied mathematics, as numerical analysis, calculus of finite differences, interpolation theory and probability theory. Two good sources are the texts by Gelfond [2] and Milne-Thomson [4].
Let $f : \mathbb{R} \to \mathbb{C}$ be a differentiable function, where $\mathbb{C}$ is the complex number. Assume that $s_0, s_1, ..., s_m$ are distinct real numbers subject to $s_0 < s_1 < \cdots < s_m$; such points are known as knots. The divided differences of f are recursively given by the following formula,[3] and [7]:

$$[f; s_0] = f(s_0),$$

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*Corresponding author

$$[f; s_0, s_1, \ldots, s_m] = \frac{[f; s_1, \ldots, s_m] - [f; s_0, s_1, \ldots, s_{m-1}]}{s_m - s_0} \quad \text{if } s_m \neq s_0.$$

From the recursive formulas, it is clear that if $s = s_0$, then

$$[f; s, s_0] = \frac{f(s) - f(s_0)}{s - s_0}.$$

If repetitions are permitted in the arguments and the function f is smooth enough, then

$$\begin{aligned} \lim_{s \to s_0} [f; s, s_0] &= \lim_{s \to s_0} \frac{f(s) - f(s_0)}{s - s_0} \\ &= f^{'}(s_0). \end{aligned}$$

This gives the definition of first-order divided difference with repeated points,

$$[f; s_0, s_0] = f^{'}(s_0).$$

In general, let $s_0 \leqslant s_1 \leqslant \cdots \leqslant s_m$; the divided differences with repeated points obey the following recursive formula:

$$[f; s_0, s_1, \cdots, s_m] = \begin{cases} \frac{[f; s_1, \cdots, s_m] - [f; s_0, s_1, \cdots, s_{m-1}]}{s_m - s_0}, & \text{if } s_m \neq s_0, \\ \frac{f^{(m)}(s_0)}{m!}, & \text{if } s_m = s_0. \end{cases} \tag{1}$$

When $s_0, s_1, \cdots, s_m$ are all distinct, it follows inductively from (1) that

$$[f; s_0, s_1, \cdots, s_m] = \sum_{k=0}^{m} \frac{f(s_k)}{v^{(1)}(s_k)}, \tag{2}$$

where $v(s) = \prod_{j=0}^{m}(s - s_j)$ and $v^{(1)}(s) = \frac{d}{ds}v(s)$, [5].

We recall that (2) is the m-th order divided differences of f on the knots $s_0, s_1, ..., s_m$.

Let us introduce the following notations. Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $m, n \in \mathbb{N}_0$, and let $k_0, k_1, \ldots, k_m, l_0, l_1, \ldots, l_n \in \mathbb{N}$ so that $k_0 + k_1 + \cdots + k_m = M + 1$ and $l_0 + l_1 + \cdots + l_n = N + 1$. Also let $\alpha = \max\{k_0 - 1, k_1 - 1, \ldots, k_m - 1\}$ and $\beta = \max\{l_0 - 1, l_1 - 1, \ldots, l_n - 1\}$; $I, J \subseteq \mathbb{R}$ be intervals, $I \times J$ be a bidimensional interval and $D^{\alpha,\beta}(I \times J)$ be the set of all real valued bivariate functions f with the property that exists $\frac{\partial f^{i+j}}{\partial s^i \partial t^j}$ on $I \times J$, where $i \in \{0, 1, \ldots, \alpha\}$ and $j \in \{0, 1, \ldots, \beta\}$.

In the following, the one dimensional divided differences with multiple knots is denoted by $[f; s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)}]$ where $s_0, s_1, \ldots, s_m$ are distinct and

$$s_j^{(k_j)} = \underbrace{s_j, s_j, \ldots, s_j.}_{k_j \ times}$$

We use $[f; s_0, s_1, \dots, s_m]$ for distinct knots only.
The formulas which are available in the literature for one dimensional divided differences with multiple knots are computationally rather complicated. It is the ratio of two determinants, namely,

$$[f(s,t); s_0^{(k_0)}, s_1^{(k_1)}, \dots, s_m^{(k_m)}]_s = \frac{(Wf)(s_0^{(k_0)}, s_1^{(k_1)}, \dots, s_m^{(k_m)})}{V(s_0^{(k_0)}, s_1^{(k_1)}, \dots, s_m^{(k_m)})}, \tag{3}$$

where

$$(Wf)(s_0^{(k_0)}, s_1^{(k_1)}, \dots, s_m^{(k_m)}) \tag{4}$$

$$= \begin{pmatrix} 1 & s_0 & s_0^2 & \cdots & s_0^{M-1} & f(s_0) \\ 0 & 1 & 2s_0 & \cdots & (M-1)s_0^{M-2} & f'(s_0) \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (M-1)(M-2)\dots.(M-k_0+1)s_0^{M-k_0} & f^{(k_0-1)}(s_0) \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 1 & s_m & s_m^2 & \cdots & s_m^{M-1} & f(s_m) \\ 0 & 1 & 2s_m & \cdots & (M-1)s_m^{M-2} & f'(s_m) \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (M-1)(M-2)\dots.(M-k_m+1)s_m^{M-k_m} & f^{(k_m-1)}(s_m) \end{pmatrix},$$

and

$$V(s_0^{(k_0)}, s_1^{(k_1)}, \cdots, s_m^{(k_m)}) \tag{5}$$

$$= \begin{pmatrix} 1 & s_0 & s_0^2 & \cdots & s_0^{M} \\ 0 & 1 & 2s_0 & \cdots & Ms_0^{M-1} \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & M(M-1)\dots.(M-k_0+2)s_0^{M-k_0+1} \\ \cdots & \cdots & & \cdots & \cdots \\ 1 & s_m & s_m^2 & \cdots & s_m^{M} \\ 0 & 1 & 2s_m & \cdots & Ms_m^{M-1} \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & M(M-1)\dots.(M-k_m+2)s_m^{M-k_m+1} \end{pmatrix},$$

is the generalized determinant of Vandermonde, [5].
Similarly, the N-th order divided differences of the function $f(s,.) : J \to \mathbb{R}$, $s \in I$, with respect to the knots $t_0^{l_0}, t_1^{l_1}, \dots, t_n^{l_n}$ is defined by

$$\left[f(s,t); t_0^{(l_0)}, t_1^{(l_1)}, \dots, t_n^{(l_n)}\right]_t = \frac{(Wf)\,(t_0^{(l_0)}, t_1^{(l_1)}, \dots, t_n^{(l_n)})}{V(t_0^{(l_0)}, t_1^{(l_1)}, \dots, t_n^{(l_n)})},$$

where $(Wf)\,(t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)})$ and $V(t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)})$ are defined as above. Soltani and Roozegar [6] give a new and compact formula for one dimensional divided differences with multiple knots. They use this formula to find the distribution of randomly weighted averages. This formula comes as a theorem in next section.

In this article we present a concise and compact formula for two dimensional divided differences with multiple knots. The advantages of our formula are in its formulation and computation. It can be readily used in calculus of divided differences, Hermite interpolation polynomial and bivariate Lagrange interpolation polynomials. As an illustration, we derive two dimensional divided differences for the $f_w(s,t) = \frac{1}{(w-s)(w-t)}$.

2. Main Results

In this section we give our formula for two dimensional divided differences with multiple knots, Theorem 2.2. First we present a compact formula for one dimensional divided differences, [6].

Theorem 2.1. *Let $f \in D^{\alpha}(I)$, where $D^{\alpha}(I)$ is the set of all real functions f, α-times differentiable on I. Then*

$$\begin{aligned}
[f; s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)}] &= \frac{(Wf)(s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)})}{V(s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)})} \\
&= \sum_{j=0}^{m} \frac{(-1)^{M+k_j-1}}{(k_j-1)!} \frac{d^{k_j-1}}{ds_j^{k_j-1}} \Big(\frac{f(s_j)}{\prod_{i=0,\, i\neq j}^{m} (s_i - s_j)^{k_i}} \Big).
\end{aligned}$$

We let $\prod_{i=0,\, i\neq j}^{m} (s_i - s_j)^{k_i} = 1$ when $m = 0$.

Now we notice to the following theorem which states the main theorem. The main difference between the following formula and the result obtained by Pop and Barbosu [5] is in its formulation, computation and application. No closed form expression for two dimensional divided differences in the case of multiple knots expressed in [5]. That formula is based on some long and complicated determinants and is used numerically in application. For instance, the formula in [5] is not applicable to find the distribution of two dimensional randomly weighted averages in [6], but the following result is a user-friendly formula for applicability and analysis.

Theorem 2.2. *Let $f \in D^{\alpha,\beta}(I \times J)$. Then the following equalities*

$$\Big[\Big[f(s,t); s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)} \Big]_s ; t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)} \Big]_t$$

$$
\begin{aligned}
&= \left[\left[f(s,t); t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)}\right]_t; s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)}\right]_s \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{(-1)^{M+N+k_i+l_j-2}}{(k_i-1)!(l_j-1)!} \frac{\partial^{k_i+l_j-2}}{\partial s_i^{k_i-1}\partial t_j^{l_j-1}} \\
&\quad \left(\frac{f(s_i,t_j)}{\prod_{q=0,q\neq i}^{m}(s_q-s_i)^{k_q}\prod_{r=0,r\neq j}^{n}(t_r-t_j)^{l_r}}\right).
\end{aligned}
$$

hold. We let $\prod_{q=0,q\neq i}^{m}(s_q-s_i)^{k_q}=1$ *when* $m=0$ *and* $\prod_{r=0,r\neq j}^{n}(t_r - t_j)^{l_r}=1$ *when* $n=0$.

Proof. For $m=n=0$, $k_0=M+1$ and $l_0=N+1$, it follows from the Definition 2.1. that

$$
\Bigg[f(s,t); \underbrace{s_0, s_0, \ldots, s_0}_{(M+1)\ times}\Bigg]_s = \frac{1}{M!}\frac{\partial^M}{\partial s_0^M} f(s_0,t),
$$

and

$$
\Bigg[f(s,t); \underbrace{t_0, t_0, \ldots, t_0}_{(N+1)\ times}\Bigg]_t = \frac{1}{N!}\frac{\partial^N}{\partial t_0^N} f(s,t_0).
$$

Therefore,

$$
\begin{aligned}
\left[\left[f(s,t); s_0^{(k_0)}\right]_s; t_0^{(l_0)}\right]_t &= \left[\frac{1}{M!}\frac{\partial^M}{\partial s_0^M} f(s_0,t); t_0^{(l_0)}\right]_t \\
&= \frac{1}{M!}\left[\frac{\partial^M}{\partial s_0^M} f(s_0,t); t_0^{(l_0)}\right]_t \\
&= \frac{1}{M!N!}\frac{\partial^{M+N}}{\partial s_0^M \partial t_0^N} f(s_0,t_0).
\end{aligned}
$$

For general m,n, by Theorem 2.1. we have

$$
\left[\left[f(s,t); s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)}\right]_s; t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)}\right]_t
$$

$$= \left[\sum_{i=0}^{m} \frac{(-1)^{M+k_i-1}}{(k_i-1)!} \frac{\partial^{k_i-1}}{\partial s_i^{k_i-1}} \left(\frac{f(s_i,t)}{\prod_{q=0,q\neq i}^{m}(s_q-s_i)^{k_q}}\right); t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)}\right]_t$$
$$= \sum_{i=0}^{m} \frac{(-1)^{M+k_i-1}}{(k_i-1)!} \left[\frac{\partial^{k_i-1}}{\partial s_i^{k_i-1}} \left(\frac{f(s_i,t)}{\prod_{q=0,q\neq i}^{m}(s_q-s_i)^{k_q}}\right); t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)}\right]_t$$
$$= \sum_{i=0}^{m} \frac{(-1)^{M+k_i-1}}{(k_i-1)!} \left(\sum_{j=0}^{n} \frac{(-1)^{N+l_j-1}}{(l_j-1)!} \frac{\partial^{l_j-1}}{\partial t_j^{l_j-1}} \left(\frac{\frac{\partial^{k_i-1}}{\partial s_i^{k_i-1}}\left(\frac{f(s_i,t_j)}{\prod_{q=0,q\neq i}^{m}(s_q-s_i)^{k_q}}\right)}{\prod_{r=0,r\neq j}^{n}(t_r-t_j)^{l_r}}\right)\right),$$

which gives the result. □

Definition 2.3. *The (m,n)-th order divided difference of the function f, where $f \in D^{\alpha,\beta}(I \times J)$, with respect to the distinct knots $(s_i,t_j) \in I \times J$, $i \in \{0,1,\ldots,m\}$ and $j \in \{0,1,\ldots,n\}$ is defined by*

$$\left[f;\ \begin{matrix} s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)} \\ t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)} \end{matrix}\right]$$
$$= \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{(-1)^{M+N+l_j+k_i-2}}{(k_i-1)!(l_j-1)!} \frac{\partial^{k_i+l_j-2}}{\partial s_i^{k_i-1}\partial t_j^{l_j-1}}$$
$$\left(\frac{f(s_i,t_j)}{\prod_{q=0,q\neq i}^{m}(s_q-s_i)^{k_q}\prod_{r=0,r\neq j}^{n}(t_r-t_j)^{l_r}}\right).$$

The following corollary has given by [1].

Corollary 2.4. *If $f(x,y) = g(x)h(y)$, $(\forall)(x,y) \in [a,b]\times[c,d]$ where $g : [a,b] \to R$, $h : [c,d] \to R$, the following equality*

$$\left[f;\ \begin{matrix} x_0, x_1, \ldots, x_m \\ y_0, y_1, \ldots, y_n \end{matrix}\right] = [g; x_0, x_1, \ldots, x_m][h; y_0, y_1, \ldots, y_n],$$

holds.

Now, we present an example using the Theorem 2.2, Corollary 2.1 and the work of Soltani and Roozegar [6].

Example 2.5. Let $f_w(s,t) = \frac{1}{(w-s)(w-t)}$ and for given real numbers $s_0, s_1, \ldots, s_m$, $t_0, t_1, \ldots, t_n, w$ and integers $k_i \geqslant 1$, $i = 0,1,\ldots,m$; $l_i \geqslant 1$, $i = 0,1,\ldots,n$ then

$$\left[f;\ \begin{matrix} s_0^{(k_0)}, s_1^{(k_1)}, \ldots, s_m^{(k_m)} \\ t_0^{(l_0)}, t_1^{(l_1)}, \ldots, t_n^{(l_n)} \end{matrix}\right] = \prod_{j=0}^{m} \frac{1}{(w-s_j)^{k_j}} \prod_{i=0}^{n} \frac{1}{(w-t_i)^{l_i}}.$$

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References

[1] D. Barbosu, Two dimensional divided differences revisited, *Creative Math. & Inf.*, 17 (2008), 1-7.

[2] A. O. Gelfond, *Calculus of Finite Differences*, Hindustan publishing corporation, Delhi-7, India, 1971.

[3] S. Karlin, C. A. Micchelli, and Y. Rinott, Multivariate splines: A probabilistic perspective. *J. Multivariate Anal.*, 20(1) (1986), 69-90.

[4] L. M. Milne-Thomson, *The Calculus of Finite Differences*, Mcmillan, New York, 1965.

[5] O. T. Pop and D. Barbosu, Two dimensional divided differences with multiple knots, *An. Şt. Univ. Ovidius Constanţa*, 17 (2009), 181-190.

[6] A. R. Soltani and R. Roozegar, On distribution of randomly ordered uniform incremental weighted averages: Divided difference approach. *Statist. Probab. Lett.*, 82(5) (2012), 1012-1020.

[7] A. Xu and C. Wang, Some extensions of fa'a di brunos formula with divided differences. *Comput. Math. Appl.*, 59(6) (2010), 2047-2052.

Rasool Roozegar
Department of Statistics
Assistant Professor of Statistics
Yazd University
P.O. Box 89195-741
Yazd, Iran
E-mail: rroozegar@yazd.ac.ir

Ali Akbar Jafari
Department of Statistics
Assistant Professor of Statistics
Yazd University
P.O. Box 89195-741
Yazd, Iran
E-mail: aajafari@yazd.ac.ir